EPiC Series in Computing
Volume 94, 2023, Pages 164–180
Proceedings of 24th International Conference on Logic for Programming, Artificial Intelligence and Reasoning

Syntactic computation of Fagin-Halpern conditioning in possibility theory

Omar Et-Targuy[1,2,3,4], Ahlame Begdouri[2], Salem Benferhat[1], Carole Delenne[3,4]

[1] CRIL, CNRS UMR 8188, University of Artois, France.
[2] Université Sidi Mohamed Ben Abdellah (USMBA), Fez, Morocco.
[3] HSM, University of Montpellier, CNRS, IRD, Montpellier, France
[4] Inria, Team Lemon, Montpellier, France

Abstract

Conditioning plays an important role in revising uncertain information in light of new evidence. This work focuses on the study of Fagin and Halpern (FH-)conditioning in the context where uncertain information is represented by weighted or possibilistic belief bases. Weighted belief bases are extensions of classical logic belief bases where a weight or degree of belief is associated with each propositional logic formula. This paper proposes a characterization of a syntactic computation of the revision of weighted belief bases (in the light of new information) which is in full agreement with the semantics of the FH-conditioning of possibilistic distributions. We show that the size of the revised belief base is linear with respect to the size of the initial base and that the computational complexity amounts to performing $\mathcal{O}(\log_2(n))$ calls to the propositional logic satisfiability tests, where n is the number of different degrees of certainty used in the initial belief base.

1 Introduction

Belief revision [15, 20] is an important problem in knowledge representation and artificial intelligence. It consists in revising plausible beliefs of an agent in the light of new information, often considered to be completely certain and reliable. If the new information to be inserted is consistent with the a priori beliefs, then the revision comes down to simply add this information to the agent's beliefs. The problem arises when this new information contradicts a priori beliefs. In this case, the agent must decide which information to ignore or replace with other weaker, less informative and less precise information.

Within the framework of uncertainty theories, the process of belief revision is materialized by the notion of conditioning. Agents' beliefs are modeled by uncertainty distributions μ (probability distribution, mass function [21, 1], possibility distribution [12], ordinal conditional function OCF [22, 23], etc.) which associate to each element of the universe of discourse (in our context, a set of propositional logic interpretations) a degree of plausibility.

In probability theory, Bayesian conditioning is widely used, especially in Bayesian networks, for the propagation of beliefs in the presence of new observations.

For belief functions, Dempster's rule of conditioning [21, 16] remains the reference operator for the revision of uncertain information.

In possibility theory, conditioning is defined in two different approaches, depending on how the uncertainty degrees are interpreted [12]. The first definition is the so-called product-based possibilistic conditioning which makes full use of the uncertainty interval $[0, 1]$. The second definition is called min-based possibilistic conditioning where only the relative order between uncertain information matters.

These standard conditioning operators (probabilistic Bayesian conditioning, Dempster's rule of conditioning, possibilistic min-based and product-based conditionings) have been very well studied in the literature, both from representational and computational points of view.

Another form of conditioning, called Fagin and Halpern conditioning (noted FH-conditioning in the following), is however little considered in the literature, in particular from a computational point of view. This conditioning was introduced in the framework of belief functions theory in [13]. FH-conditioning allows to obtain a better characterization of the conditioned belief function than Dempster's in the context of belief functions being interpreted as lower and upper probabilities, induced by a particular family of probability distributions (see also [6] for a discussion whether FH-conditioning should be considered as a revision or prediction operator). Besides, still within the framework of the theory of belief functions, other theoretical justifications of FH-conditioning have been proposed in [7, 18] and recent work have illustrated their use for object detection in the context of autonomous vehicles [19].

In the context of possibility theory, FH-conditioning has been approached only from a semantic point of view [9] and there is no work (to the best of our knowledge) that treats this conditioning from the point of view of syntactic representation of uncertain beliefs. In this paper, we are interested in addressing this shortcoming by studying FH-conditioning in a framework where uncertain beliefs are represented by sets of weighted propositional logical formulas. In possibility theory, the available uncertain information is represented by a so-called possibilistic belief base, denoted by Σ in the following. This weighted belief base is composed of a set of pairs (ϕ_i, α_i) where ϕ_i is a propositional logic formula, and α_i is a minimum degree of certainty (more precisely a degree of necessity in the sense of the possibility theory) associated with ϕ_i.

The question considered in this paper is how to revise a possibilistic weighted base Σ, in the light of a new totally certain information (denoted in this paper by $(\psi, 1)$, where ψ is a propositional logic formula) while being in full agreement with the possibilistic semantics of FH-conditioning. To achieve this goal, we first propose an equivalent syntactic reformulation of the semantic definition of FH-conditioning as a sequence of three transformation operations of possibility distributions. For each of these semantic transformation operations, we propose their equivalent characterisation on the weighted belief bases. At the end of the third operation, we show that the final belief base obtained corresponds exactly to the application of FH-conditioning on weighted belief bases. We provide at the end of the paper the spatial and temporal complexity analysis of the syntactic computation of FH-conditioning. We show that the size of the revised belief base is linear with respect to the size of the initial base. Moreover, the complexity of computing the FH-conditioning, from the weighted belief bases, comes down to the complexity of computing the degree of certainty of the new information ψ from the initial base Σ, which is done in $\log_2(n)$ calls to a satisfiability test of a set of propositional formulas (SAT).

The rest of the paper is organized as follows. Section 2 recalls possibilistic logic and presents weighted belief bases and their semantics. Section 3 introduces the notion of FH-conditioning from a semantic point of view. Section 4 contains the computation of the syntactic revision of the weighted belief bases. The last section concludes the paper.

2 A brief review of possibilistic logic

Possibilistic logic [17, 8] is a simple extension of a propositional logic. At the syntactical level, instead of considering that formulas of the propositional belief base have the same level of priority, we associate to each formula a degree which reflects its level of priority with respect to the other formulas in the belief base. The result is called a *possibilistic* belief base. The possibilistic logic semantics is also an extension of the one of propositional logic, which partitions the set of interpretations in two parts: the models and counter-models of the belief base. In possibilistic logic, we get a more refined partition, called a *possibility distribution*, where the counter-models of lower priority information will be preferred to counter-models of higher priority information. The notions of consistency and inference of propositional logic become gradual in possibilistic logic and give respectively the notions of possibility and necessity measures. Possibilistic logic ahas a connection with modal logic with points in common, especially on the duality of necessity and possibility, and differences; in particular possibilistic logic is gradual and nonmonotonic while modal logic is non-gradual and monotonic (see [2] for a more details). Possibility theory and possibilistic logic have very close links with Conditional Ordinal Functions (OCF), also called ranking functions, introduced by Spohn [22, 23]. OCFs are models widely used to represent the epistemic states of agents and to define belief revision methods.

In the following, we consider a finite propositional language $\mathcal{L}$ whose formulas are denoted by Greek letters (except for Ω and ω). Ω is the finite set of interpretations of $\mathcal{L}$ and ω an element of Ω. We denote by $\models$ the satisfaction relation of propositional logic.

2.1 Possibility distributions

A possibility distribution π is a mapping from the set of interpretations Ω to the interval $[0, 1]$. $\pi(\omega)$ represents the degree of consistency (or compatibility) of the interpretation ω with the available knowledge. By convention:

- $\pi(\omega) = 1$ means that it is entirely possible for ω to be the real world.
- $\pi(\omega) = 0$ means that ω is certainly not the real world.
- $\pi(\omega_1) > \pi(\omega_2)$ simply means that ω_1 is a preferred candidate than ω_2 for being the real state of the world.

From a possibility distribution π, we can define the degree of consistency (or possibility) and the degree of certainty (or of necessity) of a formula ϕ:

- $\Pi(\phi) = \max\{\pi(\omega)|\omega \models \phi\}$ evaluates to what extend ϕ is consistent with the available knowledge expressed by π.
- $N(\phi) = 1 - \Pi(\neg\phi)$ is used to measure to what extent a proposition ϕ is entailed by the knowledge expressed by π.

A possibility distribution π is said to be *normalized* if there exists an interpretation which is completely possible (i.e. $\exists\omega \in \Omega|\pi(\omega) = 1$).

2.2 Possibilistic belief bases

At a syntactic level, uncertain information is represented by a prioritized or weighted propositional logic knowledge bases, called *possibilistic belief bases*. A possibilistic belief base is a compact representation of a possibility distribution. Indeed, it is represented by a finite set of weighted formulas (the higher the weight, the more certain the formula) $\Sigma = \{(\phi_i, \alpha_i), i = 1, ..., n\}$, where ϕ_i is an element of $\mathcal{L}$, and $\alpha_i \in]0, 1]$ is the weight, considered as a lower bound of the degree of necessity $N(\phi_i)$. Note that formulas with $\alpha_i = 0$ are note explicitly represented in the belief base; namely only beliefs which are somewhat accepted by the agent are explicitly represented in the possibilistic belief base.

Possibilistic belief bases are compact representations of possibility distributions. Indeed, one can associate with any possibilistic belief base Σ a possibility distribution, denoted by π_Σ. Recall that the degree of possibility of an interpretation is its degree of compatibility or consistency with the belief base. Consider the case where a possibilistic belief base contains a single formula $\{(\phi, \alpha)\}$. If an interpretation ω satisfies the formula ϕ then this interpretation is completely consistent with ϕ, hence $\pi_\Sigma(\omega) = 1$. If a given interpretation ω does not satisfy the formula ϕ then $\pi_\Sigma(\omega)$ must be such that the greater the degree of certainty α of ϕ, less $\pi_\Sigma(\omega)$ is considered possible. In particular, if ϕ is completely certain, i.e. $\alpha = 1$, then ω is completely impossible, i.e. $\pi_\Sigma(\omega) = 0$.

More generally, if Σ contains more than one formula then interpretations satisfying all beliefs will have the greatest degree of possibility, i.e. 1, and the other interpretations will be ranked with respect to the belief of greater weight that they falsify. More formally:

Definition 1. *The possibility distribution associated with a possibistic belief base Σ is defined, for all $\omega \in \Omega$ by [8]:*

$$\pi_\Sigma(\omega) = \begin{cases} 1 & \text{if } \forall (\phi_i, \alpha_i) \in \Sigma, \omega \models \phi_i \\ 1 - \max\{\alpha_i : (\phi_i, \alpha_i) \in \Sigma, \omega \not\models \phi_i\} & \text{otherwise} \end{cases} \quad (1)$$

It is easy to check that the possibility distribution π_Σ is normalized if and only if the set of the non-weighted propositional logic formulas obtained from Σ, i.e. $\{\phi_i : (\phi_i, \alpha_i) \in \Sigma\}$, is consistent in the sense of propositional logic.

In the following, we will use the following academic example to illustrate the different notions of the paper:

Example 1. *Let $\Sigma = \{(\neg q \vee s, 0.66), (q \vee \neg s, 0.70), (\neg q \vee \neg r, 0.15), (q \vee s, 0.35)\}$ be a possibilistic belief base. Table 1 gives the possibility distribution associated with Σ using Equation 1.*

The interpretation $q\neg rs, qr$ is the most preferred one since it is the only one which is consistent with Σ. Hence, their possibility degree is 1. The interpretation qrs gets the possibility degree 0.85, because it falsifies the least certain belief in Σ; namely $(\neg q \vee \neg r, 0.15)$. The two following interpretations $\neg qr\neg s$ and $\neg q\neg r\neg s$ get the possibility degree 0.65, because they falsify $(q \vee s, 0.35)$. Then the next more plausible interpretations are $qr\neg s$ and $q\neg r\neg s$. Finally, the least preferred interpretations are $\neg qrs$ and $\neg q\neg rs$, as they falsify the highest belief in Σ, namely $(q \vee \neg s, 0.70)$.

q	r	s	$\pi_\Sigma(\omega)$
1	1	1	0.85
1	1	0	0.34
1	0	1	1
1	0	0	0.34
0	1	1	0.3
0	1	0	0.65
0	0	1	0.3
0	0	0	0.65

Table 1: The possibility distribution associated with Σ

3 Semantic definition of Fagin-Halpern conditioning

This section concerns the problem of beliefs changes, in possibility theory, which is an important topic in artificial intelligence and in the information systems for managing the dynamics of beliefs.
At the semantic level, the revision of a possibility distribution π consists in changing the plausibility order between the interpretations in order to give priority to new information, denoted ψ. This revision process is obtained thanks to the notion of *conditioning* which transforms a distribution of possibility *a priori* π and a certain information ψ into a new possibility distribution (a posteriori) denoted by $\pi(.|\psi)$.

The possibility distribution $\pi(.|\psi)$ must be normalised or consistent (i.e., $\exists\omega \in \Omega$ such that $\pi(\omega \mid \psi) = 1$)) and where the new information ψ should be completely certain (i.e., $\forall\omega \in \Omega$ such that ω falsifies ψ, we have $\pi(\omega \mid \psi) = 0$)). Besides, the new revised possibility distribution $\pi(.|\psi)$ must not change the relative order between the models of ψ (i.e., $\forall\omega \in \Omega$ and $\forall\omega' \in \Omega$ such that $\omega \models \psi$ and $\omega' \models \psi$, if $\pi(\omega) \geq \pi(\omega')$ then $\pi(\omega \mid \psi) \geq \pi(\omega' \mid \psi)$). Even if in this paper we use the minimum and the maximum operators (particular cases of t-norms and t-conorms), this work mainly deals with uncertainty and the case of vague or fuzzy information [24] is not treated in this paper.

Clearly, there are different ways to define $\pi(. \mid \psi)$ (e.g., [5]). Two main definitions have been proposed within the framework of possibility theory (e.g., [11]). The first definition, called min-based possibilistic conditioning, consists in i) setting the countermodels of ψ (the new information) to 0, ii) setting the best models of ψ to 1, and iii) letting the degrees of possibility of the other models of ψ unchanged. The second definition, called product-based possibilistic conditioning, consists in i) setting the countermodels of ψ (the new information) to 0, and ii) proportionally shifting the degrees of possibility of the models of ψ up to to obtain a normalized conditional possibility distribution.

These two definitions are formally described as follows (when $\Pi(\phi) > 0$):

- Minimum-based conditioning:

$$\pi(\omega \mid_m \psi) = \begin{cases} 1, & \text{if } \pi(\omega) = \Pi(\psi) \text{ and } \omega \models \psi \\ \pi(\omega), & \text{if } \pi(\omega) < \Pi(\psi) \text{ and } \omega \models \psi \\ 0, & \text{if } \omega \not\models \psi \end{cases} \tag{2}$$

q	r	s	$\pi_\Sigma(\omega \mid_{FH} q \vee \neg r \vee s)$
1	1	1	0.85
1	1	0	0.493
1	0	1	1
1	0	0	0.493
0	1	1	0.462
0	1	0	0
0	0	1	0.462
0	0	0	0.65

Table 2: FH-conditioning of the possibility distribution of Example 1

- Product-based conditioning:

$$\pi(\omega \mid_p \psi) = \begin{cases} \frac{\pi(\omega)}{\Pi(\psi)}, & \text{if } \omega \models \psi \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

There is an alternative to these two definitions which has been very little explored in possibility theory. This is the conditioning proposed by Fagin and Halpern [14] and introduced within the framework of belief functions. This definition of conditioning, which we will denote hereinafter by FH-conditioning, has been adapted to possibility theory and was directly defined on possibility measures (instead of possibility distributions) as follows [9]):

$$\Pi(\phi \mid_{FH} \psi) = \frac{\Pi(\phi \wedge \psi)}{\Pi(\phi \wedge \psi) + N(\neg\phi \wedge \psi)} \tag{4}$$

We recall that in the above equation $N(\phi) = 1 - \Pi(\neg\phi)$. Now, if we restrict the definitions $\Pi(. \mid_{FH} \psi)$ to interpretations, we get the definition of FH-conditioning defined on possibility distributions [9].

Definition 2. *Let π be a possibility distribution and let ψ be a propositional logic formula. Then the FH-conditioning of π by ψ, denoted by $\pi(. \mid_{FH} \psi)$, is given by the following equation:*

$$\forall \omega \in \Omega, \quad \pi(\omega \mid_{FH} \psi) = \begin{cases} \max\left(\pi(\omega), \frac{\pi(\omega)}{\pi(\omega)+N(\psi)}\right) & if \omega \models \psi \\ 0 & if \omega \not\models \psi \end{cases} \tag{5}$$

Where again we recall that

$$N(\phi) = 1 - \Pi(\neg\phi),$$

with

$$\Pi(\psi) = \max\{\pi_\Sigma(\omega) : \omega \in \Omega \text{ and } \omega \models \psi\}.$$

In the rest of the paper, and when there is no ambiguity, we will simply use $\pi(. \mid \psi)$ instead of $\pi(. \mid_{FH} \psi)$. Let us continue our example.

Example 2. *Let us apply the possibilistic FH-conditioning at the semantic level to the possibility distribution π_Σ given in the example 1 by the new piece of information $q \vee \neg r \vee s$. Table 2 gives the FH-conditioned possibility distribution.*

Let us explain the result of FH-conditioning with the input propositional logic formula $q \vee \neg r \vee s$.
The interpretation $\neg q r \neg s$ is not model of $q \vee \neg r \vee s$. Therefore, its conditional possibility degree is equal 0.
The others interpretations are models of $q \vee \neg r \vee s$. Therefore, after conditioning, their possibility degree will be equal to $\max\left(\pi(\omega), \frac{\pi(\omega)}{\pi(\omega)+N(\psi)}\right)$.

4 Syntactic counterpart of possibilistic Fagin-Halpern conditioning

There is a clear interest in FH-conditioning for the revision of uncertain information within the framework of belief functions theory, both at the theoretical level and at the level of applications.
In the framework of possibility theory, the very few works that exist (e.g., [9]) mainly concern semantic definitions of FH-conditioning (e.g., defined on the set of interpretations of propositional logic) or on discussions whether the FH-conditioning is a revision operator, or a prediction operator or what is called a focusing operator (see [10] and [6] for more details).

However, unlike min-based and product-based conditioning (and also their hybrid version [3]) which have been well studied in the literature, there is no work (to the best of our knowledge) that deals with syntactic computation of possibilistic FH-conditioning applied on possibilistic belief bases.

Defining a syntactic FH-conditioning, i.e. at the level of possibilistic knowledge bases, has clear computational advantages over an application of FH-conditioning at the semantic level, since obviously the size of the set interpretations is exponential with respect to the number of propositional symbols of the used language.

Figure 1 summarizes the purpose of this section and is explained as follows:

- Inputs of our syntactic FH-conditioning computation are:
  - a possibilistic belief base Σ; and
  - a totally certain new piece of information $(\psi, 1)$ that must be taken into account.
- Let π_Σ be the possibility distribution associated with Σ using Equation 1.
- Let $\pi_\Sigma(.|\psi)$ be the conditional possibility distribution, resulting from the semantic application of FH-conditioning on π_Σ after the integration of the new computing $(\psi, 1)$ using the Equation 5.
- Our goal is to compute, from Σ and the totally certain information $(\psi, 1)$, a new possibilistic belief base, denoted Σ_{FH}, such that:

$$\forall \omega \in \Omega, \pi_{\Sigma_{FH}}(\omega) = \pi_\Sigma(\omega \mid \psi). \tag{6}$$

  where $\pi_{\Sigma_{FH}}$ is the possibility distribution associated with Σ_{FH} using the equation 1.

The satisfaction of Equation 6 is necessary for having a syntactic computation that fully agrees with the semantic definition of FH-conditioning in possibility theory.

The following subsection first gives an equivalent decomposition of the definition of FH-conditioning into three elementary steps of transforming possibility distributions. Then, we give the syntactic counterpart of each of these three steps; before finally putting together all the syntactical computations to achieve the objective of this section.

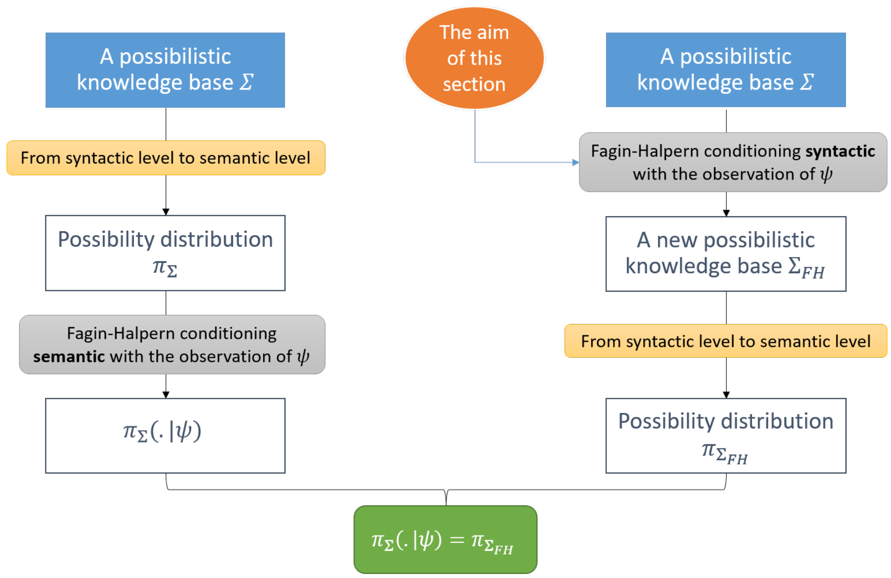

Figure 1: The aim of this section

4.1 Possibilistic FH-conditioning in three main steps

The computation of the syntactic counterpart of Equation 5 is not straightforward. We are therefore going to describe it progressively in an alternative form by breaking down the semantic definition of FH-conditioning (given by Equation 5) into three elementary steps of modification of possibility distributions.

Let π_Σ be the initial possibility distribution associated with the initial possibilistic belief base Σ (using Equation 1). Let $(\psi, 1)$ be the new piece of information that should be taken into account. We define the three steps as follows:

- **Step 1: shifting the initial possibility distribution.** This step mainly consists of shifting each possibility degree $\pi_\Sigma(\omega)$ by integrating the necessity degree associated with the new information. The result of this modification of the initial possibility distribution is simply denoted π_1 and is formally defined as follows:

$$\pi_1(\omega) = \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega) + N(\psi)},$$

where we recall that $N(\psi) = \max\{\pi_\Sigma(\omega) : \omega \in \Omega \textit{ and } \omega \models \psi\}$.

- **Step 2: combining possibility distributions.** This second step consists in combining with the maximum operator, the result of the possibility distribution of step 1 with the initial possibility distribution π_Σ.

  The possibility distribution obtained at this stage is again simply denoted by π_2 and is defined as follows:

$$\pi_2(\omega) = \max(\pi_\Sigma(\omega), \pi_1(\omega)).$$

- **Step 3: integrating new information.** This third and final step consists simply of integrating the new information $(\psi, 1)$ into the possibility distribution obtained in step 2. For this, we simply encode the new information $(\psi, 1)$ by a binary possibility distribution, where only possibility degree 1 (totally possible) and possibility degree 0 (totally impossible) are used. The possibility degree 1 is associated with models of ψ while the possibility degree 0 is associated with its countermodels.

  The possibility distribution obtained in step 3 is again simply denoted by π_3 and is defined as:
$$\pi_3(\omega) = \min(\pi_2(\omega), \pi_\psi(\omega)),$$
  where,
$$\pi_\psi(\omega) = \begin{cases} 1, & \omega \models \psi, \\ 0, & otherwise. \end{cases}$$

It is not difficult to show, that at the end of the third step, the final obtained possibility distribution is exactly that resulting from applying FH-conditioning on π_Σ with $(\psi, 1)$ and given by Equation 5; i.e. we have:

$$\forall \omega \in \Omega, \pi_3(\omega) = \pi(\omega \mid_{FH} \psi).$$

Thus, finding the syntactic counterpart of Equation 5 is equivalent to finding the syntactic counterpart of π_3.

The following three subsections give the syntactic counterparts of the possibility distributions obtained at each of the three steps described above (Subsection 4.1).

4.1.1 Syntactic computation of Step 1: shifting the initial possibility distribution

We begin by giving the syntactic counterpart of the possibility distribution obtained in Step 1 and denoted by π_1. Proposition 1 provides a possibilistic belief base, denoted by Σ_1, which compactly encodes the distribution π_1. More precisely, the resulting possibilistic belief base Σ_1 is composed of two parts. The first part, denoted in Proposition 1 by Σ_{11}, consists in modifying the possibility degrees associated with each of the formulas of the initial base Σ, in order to reflect the shift made in this step 1 on the initial possibility distribution. The second part is composed of a single formula, which is a weighted contradiction, that expresses the fact that π_1 can be sub-normalized (i.e. there can be no interpretation ω, such as $\pi_1(\omega) = 1$)). We will see in the computation of the possibilistic belief base associated with the Step 2 that this sub-normalization will be lifted and the result obtained at this step 2 is always normalized and consistent.

The precise result of the computation of the possibilistic belief base associated with the possibility distribution of step 1 is given in the following Proposition 1.

Proposition 1. *Let $\Sigma = \{(\phi_i, \alpha_i), i = 1, ..., n\}$ be the initial belief base. Let $(\psi, 1)$ be the new piece of information. Let $\Sigma_1 = \{(\perp, 1 - \frac{1}{1+N(\psi)})\} \cup \Sigma_{11}$ be a belief base obtained from Σ, with $\Sigma_{11} = \{(\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}), (\phi_i, \alpha_i) \in \Sigma\}$. Then, $\forall \omega \in \Omega$, we have:*

$$\pi_{\Sigma_1}(\omega) = \pi_1(\omega) = \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega) + N(\psi)}, \tag{7}$$

where $\pi_{\Sigma_1}(\omega)$ is the possibility distribution associated with Σ_1 and π_1 is the possibility distribution obtained at Step 1 described in subsection 4.1.

Proof. Let $\omega \in \Omega$. There are two cases to consider:

1. ω is a model of all formulas of Σ. Namely, $\pi_\Sigma(\omega) = 1$. In this case, ω is also a model of all formulas of Σ_1. Therefore:

$$\begin{aligned}\pi_{\Sigma_1}(\omega) &= 1 - (1 - \frac{1}{1+N(\psi)}) \\ &= \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega)+N(\psi)}.\end{aligned}$$

2. ω falsifies at least a weighted formula from Σ. This also means that ω falsifies at least a weighted formula from Σ_1. In this case:

$$\begin{aligned}\pi_{\Sigma_1}(\omega) &= \min\{1 - (1 - \frac{1}{1+N(\psi)}), \pi_{\Sigma_{11}}(\omega)\} \\ &= \min\{\frac{1}{1+N(\psi)}, \pi_{\Sigma_{11}}(\omega)\} \\ &= \pi_{\Sigma_{11}}(\omega).\end{aligned}$$

Indeed, $\pi_{\Sigma_{11}}(\omega)$ is necessarily of the form $\frac{1-\alpha_i}{1-\alpha_i+N(\psi)}$ for some $(\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)})$ in Σ_{11}. And we have $\frac{1-\alpha_i}{1-\alpha_i+N(\psi)} < \frac{1}{1+N(\psi)}$ (we recall that $\alpha_i > 0$). Now, we develop the expression associated with π_{Σ_1}:

$$\begin{aligned}\pi_{\Sigma_1}(\omega) &= \pi_{\Sigma_{11}}(\omega) \\ &= \min\{1 - [1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}] : (\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}) \in \Sigma_{11} \textit{ and } \omega \not\models \phi_i\} \\ &= \min\{\frac{1-\alpha_i}{1-\alpha_i+N(\psi)} : (\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}) \in \Sigma_{11} \textit{ and } \omega \not\models \phi_i\} \\ &= \frac{\min\{1-\alpha_i : (\phi_i, \alpha_i) \in \Sigma \textit{ and } \omega \not\models \phi_i\}}{\min\{1-\alpha_i+N(\psi) : (\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}) \in \Sigma_{11} \textit{ and } \omega \not\models \phi_i\}} \\ &= \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega)+N(\psi)}.\end{aligned}$$

□

The following example 3 illustrates the computation of the possibilistic belief base given in the Proposition 1 on our running example.

Example 3. *Let us consider the possibility distribution $\pi_\Sigma(\omega)$ of Example 1. Let $\psi = q \vee \neg r \vee s$ be the new piece of information. Note that from Table 1, $\Pi(\neg q \wedge r \wedge \neg s) = 0.65$. Hence, $N(q \vee \neg r \vee s) = 0.35$ The following table gives the values of:*

$$\pi_1(\omega) = \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega)+N(q \vee \neg r \vee s)}.$$

Let us compute the belief base Σ_1 associated with π_1.

q	r	s	$\pi_1(\omega)$
1	1	1	0.708
1	1	0	0.493
1	0	1	0.74
1	0	0	0.493
0	1	1	0.462
0	1	0	0.65
0	0	1	0.462
0	0	0	0.65

Table 3: The possibility distribution $\pi_1(\omega)$

From Proposition 1, the general form of Σ_1 is:

$\Sigma_1 = \{(\perp, 1 - \frac{1}{1+N(\psi)})\} \cup \Sigma_{11}$, *with* $\Sigma_{11} = \{(\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}), (\phi_i, \alpha_i) \in \Sigma\}$.
Replacing ψ by $q \vee \neg r \vee s$, we get:
$\Sigma_1 = \{\perp, 0.26\} \cup \{(\neg q \vee s, 0.507), (q \vee \neg s, 0.538), (\neg q \vee \neg r, 0.292), (q \vee s, 0.35)\}$.
One can easily check that if we compute the possibility distribution π_{Σ_1} associated with the possibilitic belief base Σ_1, using Equation 1, we get exactly the one given in the above table, which means that $\forall \omega \in \Omega, \pi_{\Sigma_1}(\omega) = \pi_1(\omega)$.

4.2 Syntactic computation of Step 2: combining possibility distributions

We now present the syntactic computation of Step 2. It consists of the computation of a possibilistic belief base, which we will denote Σ_2, whose associated distribution is equal to that obtained from step 2 and denoted by π_2.
We recall that π_2 is the maximum of π_1 and the possibility distribution π_Σ associated with the initial belief base Σ. There exists in the literature (e.g., [4]) a characterization of the combination by the maximum operator. However, such a characterization would yield a resultant possibilistic belief base of size equal to the product of the two sizes of the belief bases to be combined (in our case in $\mathcal{O}(\mid \Sigma \mid) \times (\mid \Sigma_1 \mid))$, where $\mid \Sigma \mid$ (respectively $\mid \Sigma_1 \mid$) is the number of weighted formulas in Σ (respectively in Σ_1)). Proposition 2 gives a better spatial complexity result. Indeed, we show that the resulting base, associated with the syntactic combination based on the maximun of π_1 and π_Σ, is of linear size with respect to those of the possibilistic belief bases to be combined; and more precisely in $\mathcal{O}(\max(\mid \Sigma \mid), \mid \Sigma_1 \mid)))$.
Besides, as π_Σ is normalized then trivially the distribution π_2 (defined as being the maximum of the distributions π_Σ and π_1) is also normalized. As a result, the weighted contradiction formula that was present in the resulting possibilistic belief base of step 1, will no longer be present in the belief base obtained at the end of step 2.
In summary, the computation of the possibilistic belief base associated with the possibility distribution of stage 2 is given in the following Proposition 2.

Proposition 2.

- *Let $\Sigma = \{(\phi_i, \alpha_i), i = 1, ..., n\}$ be the initial belief base and ψ be the new piece of information.*
- *Let Σ_1 be the belief base obtained from proposition 1; namely,* $\Sigma_1 = \{(\perp, 1 - \frac{1}{1+N(\psi)})\} \cup \Sigma_{11}$, *with* $\Sigma_{11} = \{(\phi_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}), (\phi_i, \alpha_i) \in \Sigma\}$.

- *Let* $\Sigma_2 = \{(\phi_i, \min(\alpha_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)})), (\phi_i, \alpha_i) \in \Sigma\}$ *be a belief base obtained from* Σ.

Then, $\forall \omega \in \Omega$, *we have:*

$$\begin{aligned}\pi_{\Sigma_2}(\omega) &= \pi_2(\omega)\\ &= \max\left(\pi_\Sigma(\omega), \pi_{\Sigma_1}(\omega)\right)\\ &= \max\left(\pi_\Sigma(\omega), \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega)+N(\psi)}\right),\end{aligned}$$

where $\pi_\Sigma, \pi_{\Sigma_1}$ *and* π_{Σ_2} *denote the possibility distributions associated with the belief's bases* Σ, Σ_1 *and* Σ_2 *respectively. The distribution* π_2 *is the one obtained at Step 2 and described in subsection 4.1.*

Proof. Let $\omega \in \Omega$. There are two cases to consider:

1. ω is a model of all formulas of Σ. Namely, $\pi_\Sigma(\omega) = 1$. In this case:

$$\pi_{\Sigma_1}(\omega) = \frac{1}{1+N(\psi)}, \text{ and } \pi_{\Sigma_2}(\omega) = 1.$$

   Therefore, we indeed have: $\pi_{\Sigma_2}(\omega) = \max\left(\pi_\Sigma(\omega), \pi_{\Sigma_1}(\omega)\right) = 1$.

2. ω falsifies at least a weighted formula from Σ. For sake of simplicity, we note:

   - π, π_1, and π_2 instead of $\pi_\Sigma, \pi_{\Sigma_1}$, and π_{Σ_2}.
   - β_i instead of $1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)}$.
   - m the number of formulas from Σ (resp Σ_1) falsified by ω. Note that, by construction, Σ and Σ_1 contain exactly the same formulas. However, they differ on the weights associated with these formulas. Hence, m is the same for both Σ and Σ_1.
   - $\{\alpha_1, ..., \alpha_m\}$ and $\{\beta_1, ..., \beta_m\}$ be the sets of weights associated with falsified formulas in Σ and Σ_2 respectively.

   By definition, we have for all $\omega \in \Omega$:

$$\begin{aligned}\pi_2(\omega) &= \min\{1 - \min(\alpha_i, \beta_i) : (\phi_i, \alpha_i) \in \Sigma \textit{ and } (\phi_i, \min(\alpha_i, \beta_i)) \in \Sigma_2\}\\ &= \min\{\max(1-\alpha_i, 1-\beta_i) : i = 1, ..., m\}.\\ &\text{(for sake of simplicity, we avoid repeting } (\phi_i, \alpha_i) \in \Sigma)\end{aligned}$$

   By distributing min over the max, we obtain:
$$\pi_2(\omega) = \max\{\min\{x_1, ..., x_m\} : x_i \in \{1-\alpha_i, 1-\beta_i\}, i = 1, ..., m\}$$
   Note that when all $x_i's$ are respectively equal to $(1-\alpha_i)'s$ then: $\min\{x_i, ..., x_m\} = \min\{1-\alpha_1, ..., 1-\alpha_k, ..., 1-\alpha_m\} = 1-\alpha_k = \pi(\omega)$, for some $k \in \{1, ..., m\}$.

   Similarly, when all $x_i's$ are respectively equal to $(1-\beta_i)'s$ then: $\min\{x_i, ..., x_m\} = \min\{1-\beta_1, ..., 1-\beta_t, ..., 1-\beta_m\} = 1-\beta_t = \pi_1(\omega)$, for some $t \in \{1, ..., m\}$.

   For the other cases, some $x_i's$ are equal to $(1-\alpha_i)$ while others $x_i's$ are equal to $(1-\beta_i)$, then we have three cases to consider:

   (a) $\min\{x_1, ..., x_m\} = 1-\alpha_y$ for some $y \in \{1, ..., m\}$ and $\alpha_y \neq \alpha_k$. Then $x_k = 1-\alpha_k = \pi(\omega)$, and $\pi_1(\omega) \geq \min\{x_1, ..., x_m\}$.

(b) $\min\{x_1, ..., x_m\} = 1 - \beta_z$ for some $z \in \{1, ..., m\}$ and $\beta_z \neq \beta_t$. Then $x_t = 1 - \alpha_t = \pi(\omega)$, and $\pi(\omega) \geq \min\{x_1, ..., x_m\}$.

(c) $\min\{x_1, ..., x_m\} = 1 - \alpha_y = 1 - \alpha_k = \pi(\omega)$, or $\min\{x_1, ..., x_m\} = 1 - \beta_z = 1 - \beta_k = \pi_1(\omega)$. Then trivially, $\max(\pi(\omega), \pi_1(\omega) \geq \min\{x_1, ..., x_m\})$.

Therefore for any combination $\min\{x_1, ..., x_m\}$ where there exist some $x_i's$ equal to $(1 - \alpha_i)'s$ and also some $x_j's$ equal to $(1 - \beta_i)'s$ we have: $\pi(\omega) \geq \{x_1, ..., x_m\}$ and $\pi_1(\omega) \geq \{x_1, ..., x_m\}$.

Therefore:

$$\begin{aligned}\pi_2(\omega) &= \max\{\min\{x_1, ..., x_m\} : x \in \{1 - \alpha_i, 1 - \beta_i\}, i = 1, ..., m\} \\ &= \max\{\pi(\omega), \pi_1(\omega)\}.\end{aligned}$$

□

Note that in Proposition 2, if a given $\omega \in \Omega$ is a model of all formulas of Σ, then $\pi_{\Sigma_1}(\omega) = \frac{1}{1+N(\psi)}$. However, for inteptretation $\omega \in \Omega$ that are not models of some formulas of Σ, $\pi_{\Sigma_1}(\omega) = \frac{\pi_\Sigma(\omega)}{\pi_\Sigma(\omega)+N(\psi)}$. The following Example 4 illustrates Proposition 2.

Example 4. *Let us consider the possibility distributions $\pi_\Sigma(\omega)$ and $\pi_1(\omega)$ of the examples 1 and 3 respectively. Let $\psi = q \vee \neg r \vee s$ be the new piece of information. Rcall that $N(q \vee \neg r \vee s) = 0.35$. Table 4 gives the values of $\pi_2(\omega) = \max(\pi_\Sigma(\omega), \pi_1(\omega))$.*

q	r	s	$\pi_\Sigma(\omega)$	$\pi_1(\omega)$	$\pi_2(\omega)$
1	1	1	0.85	0.708	0.85
1	1	0	0.34	0.493	0.493
1	0	1	1	0.74	1
1	0	0	0.34	0.493	0.493
0	1	1	0.3	0.462	0.462
0	1	0	0.65	0.65	0.65
0	0	1	0.3	0.462	0.462
0	0	0	0.65	0.65	0.65

Table 4: The possibility distribution $\pi_2(\omega)$

Let us compute, using Proposition 2, the belief base:

$$\Sigma_2 = \{(\phi_i, \min(\alpha_i, 1 - \frac{1-\alpha_i}{1-\alpha_i+N(\psi)})), (\phi_i, \alpha_i) \in \Sigma\}$$

for the new piece of information $\psi = q \vee \neg r \vee s$. We have:

$$\begin{aligned}\Sigma_2 = \{&(\neg q \vee s, \min(0.66, 0.507)), (q \vee \neg s, \min(0.7, 0.538)), (\neg q \vee \neg r, \min(0.15, 0.292)), \\ &(q \vee s, \min(0.35, 0.35))\} \\ = \{&(\neg q \vee s, 0.507), (q \vee \neg s, 0.538, (\neg q \vee \neg r, 0.15), (q \vee s, 0.35)\}.\end{aligned}$$

Finally, one can check that computing the possibility distribution $\pi_{\Sigma_2}(\omega)$ for the belief base Σ_2, using Equation 1, leads exactly to the one of the above table, which means that $\forall \omega \in \Omega, \pi_{\Sigma_2}(\omega) = \pi_2(\omega)$.

4.3 Syntactic computation of Step 3: integrating new information

The last step consists in integrating, in the possibility distribution obtained in step 2, the fact that the new information ψ is totally certain. The computation of the syntactic counterpart of this step is immediate as it simply amounts to adding to the possibilistic belief base, resulting from step 2, the weighted formula $(\psi, 1)$. More specifically, we have:

Proposition 3. *Let $\Sigma = \{(\phi_i, \alpha_i), i = 1, ..., n\}$ be the initial belief base. Let ψ be a new piece of information. Let Σ_2 be the belief base obtained from proposition 2. Let $\Sigma_3 = \Sigma_2 \cup \{(\psi, 1)\}$. Then, $\forall \omega \in \Omega$:*

$$\pi_{\Sigma_3}(\omega) = \begin{cases} \pi_{\Sigma_2}(\omega), & if \omega \models \psi, \\ 0, & otherwise. \end{cases} \tag{8}$$

Proof. The proof is immediate using the definition of a possibility distribution associated with a belief base. Indeed, let ω be an interpretation. If ω is not model of of ψ, then $\pi_\Sigma(\omega) = 0$ as it falsifies a fully certain formula $(\psi, 1)$. If ω is a model of ψ, then trivially $\pi_3(\omega) = \pi_2(\omega)$. □

To sum up, propositions 1-3 allow us to provide the syntactic counterpart of the possibilistic Fagin-Halpern conditioning defined by the equation 5. The final possibilistic belief base is summarized by:

$$\Sigma_{FH} = \{(\psi, 1)\} \cup \{(\phi_i, \min(\alpha_i, 1 - \frac{1 - \alpha_i}{1 - \alpha_i + N(\psi)})), (\phi_i, \alpha_i) \in \Sigma\}. \tag{9}$$

Our last example confirms that the final computation of the possibilistic belief base at the end of Step 3 represents the FH-conditioning defined on the initial distribution π_Σ.

Example 5. *Let us consider the possibility distribution $\pi_2(\omega)$ of Example 4. Recall that the new piece of information is $\psi = q \vee \neg r \vee s$. The following table gives the values of $\pi_3(\omega) = \min(\pi_2(\omega), \pi_\psi(\omega))$, where we recall that:*

$$\pi_\psi(\omega) = \begin{cases} 1, & \omega \models \psi \\ 0, & otherwise \end{cases}$$

Using Equation 9, we get:

q	r	s	$\pi_2(\omega)$	$\pi_\psi(\omega)$	$\pi_3(\omega)$
1	1	1	0.85	1	0.85
1	1	0	0.493	1	0.493
1	0	1	1	1	1
1	0	0	0.493	1	0.493
0	1	1	0.462	1	0.462
0	1	0	0.65	0	0
0	0	1	0.462	1	0.462
0	0	0	0.65	1	0.65

Table 5: The possibility distribution $\pi_3(\omega)$

$$\Sigma_{FH} = \{(q \vee \neg r \vee s, 1)\} \cup \{(\neg q \vee s, 0.507), (q \vee \neg s, 0.538), (\neg q \vee \neg r, 0.15), (q \vee s, 0.35)\}.$$

Finally, one can check that computing the possibility distribution $\pi_{\Sigma_{FH}}$ associated with the belief base Σ_{FH}, using Equation 1, gives exactly the same distribution given in the table above. This result is also identical to the possibility distribution given in Example 2 when applying the semantic FH-conditioning with $\psi = q \vee \neg r \vee s$.

We finish this section by giving the complexity results. It is easy to check from Equation 9 that the spatial complexity is linear with respect to the size of the initial base Σ and more precisely the size of the final belief base is in $\mathcal{O}(\mid \Sigma \mid)$. This very good result is obtained thanks to the decomposition of the FH-conditioning into three elementary operations. Indeed, as explained in subsection 4.2, a direct application of existing results in the literature (e.g., [4]) regarding Step 2 (the max-based combination of possibility distributions) would generate a quadratic spatial complexity with respect to the size of the initial belief base (i.e., in $\mathcal{O}(\mid \Sigma \mid^2)$). Regarding time complexity, the difficult task in Equation 9 is the computation of the degree of necessity $N(\psi)$ from the initial base. The complexity of reasoning in possibilistic logic has been well studied in the literature (e.g., [17]). The degree $N(\psi)$ is equal to the highest degree α_i such that $\{\phi_i | (\phi_i, \alpha_j) \in \Sigma \textit{ and } \alpha_i \geq \alpha_j\}$ is consistent and infers ψ, where the notions of inference and consistency are those of propositional logic. If such a degree does not exist then $N(\psi) = 0$. If we note n the number of different degrees in the possibilistic belief base Σ, then it is easy to provide a dichotomous search algorithm for calculating this largest weight α_i in $\log_2(n)$ calls to the propositional logic satisfiability test.

5 Conclusion

The contribution of this paper concerns the question of the revision of uncertain information. We focused on Fagin and Halpern conditioning adapted to possibility theory. In particular, we have proposed a method which makes it possible to calculate the result of possibilistic FH-conditioning of a weighted base in order to take into account uncertain information. We have shown that our syntactic computation is in full agreement with the semantics given by FH-conditioning on possibility distributions. This syntactic computation was done without additional cost compared to the two other forms of possibilistic conditioning (min-based and product-based conditionings). Finally, we showed that the spatial complexity of the conditioned belief base is linear with respect to the size of the initial belief base. The Fagin and Halpern conditioning was defined in the framework of belief functions. Possibility theory can be interpreted as a particular case of belief functions when the focal elements (ie the elements that have a non-zero mass) are nested (also called consonant belief functions). A future work is to see to what extent our syntactic results can be extended to belief functions, in particular when the latter are represented in a compact way by graphical models.

Acknowledgements:

This research has received support from the European Union's Horizon research and innovation programme under the MSCA-SE (Marie Skłodowska-Curie Actions Staff Exchange) grant agreement 101086252; Call: HORIZON-MSCA-2021-SE-01; Project title: STARWARS (STormwAteR and WastewAteR networkS heterogeneous data AI-driven management).
This research has also received support from the French national project ANR (Agence Nationale de la Recherche) CROQUIS (Collecte, représentation, complétion, fusion et interrogation de données hétérogènes et incertaines de réseaux d'eaux urbains) and from the ANR project EXPIDA (EXplainable and parsimonious Preference models to get the most out of Inconsistent DAtabases), grant number ANR-22-CE23-0017.

References

[1] John C. Aldrich, A. Philip Dawid, Thierry Denoeux, Prakash P. Shenoy, and Vladimir Vovk. Glenn shafer - A short biography. *Int. J. Approx. Reason.*, 141:5–10, 2022.

[2] Mohua Banerjee, Didier Dubois, Lluis Godo, and Henri Prade. On the relation between possibilistic logic and modal logics of belief and knowledge. *Journal of Applied Non-Classical Logics*, pages 1–19, 03 2018.

[3] Salem Benferhat, Celia da Costa Pereira, and Andrea Tettamanzi. Syntactic computation of hybrid possibilistic conditioning under uncertain inputs. pages 739–745, 08 2013.

[4] Salem Benferhat and Claudio Sossai. Merging uncertain knowledge bases in a possibilistic logic framework. *CoRR*, abs/1301.7359, 2013.

[5] Bernadette Bouchon-Meunier, Giulianella Coletti, and Christophe Marsala. Independence and possibilistic conditioning. *Ann. Math. Artif. Intell.*, 35:107–123, 05 2002.

[6] Thierry Denœux, Didier Dubois, and Henri Prade. *Representations of Uncertainty in AI: Beyond Probability and Possibility*, pages 119–150. Springer International Publishing, Cham, 2020.

[7] Jean Dezert, Albena Tchamova, and Deqiang Han. Total belief theorem and conditional belief functions. *Int. J. Intell. Syst.*, 33(12):2314–2340, 2018.

[8] Didier Dubois, Jérôme Lang, and Henri Prade. Handbook of logic in artificial intelligence and logic programming, tome 3: Nonmonotonic reasoning and uncertain reasoning, chapitre possibilistic logic, 1994.

[9] Didier Dubois and Henri Prade. Bayesian conditioning in possibility theory. *Fuzzy sets and systems*, 92(2):223–240, 1997.

[10] Didier Dubois and Henri Prade. Focusing vs. belief revision: A fundamental distinction when dealing with generic knowledge. 1244, 07 1997.

[11] Didier Dubois and Henri Prade. Possibilistic logic: a retrospective and prospective view. *Fuzzy Sets and Systems*, 144(1):3–23, 2004. Possibilistic Logic and Related Issues.

[12] Didier Dubois and Henry Prade. *Possibility Theory and Its Applications: Where Do We Stand?*, pages 31–60. Springer Berlin Heidelberg, Berlin, Heidelberg, 2015.

[13] Ronald Fagin and Joseph Y. Halpern. A new approach to updating beliefs. In Piero P. Bonissone, Max Henrion, Laveen N. Kanal, and John F. Lemmer, editors, *UAI '90: Proceedings of the Sixth Annual Conference on Uncertainty in Artificial Intelligence, MIT, Cambridge, MA, USA, July 27-29, 1990*, pages 347–374. Elsevier, 1990.

[14] Ronald Fagin and Joseph Y Halpern. A new approach to updating beliefs. *arXiv preprint arXiv:1304.1119*, 2013.

[15] Peter Gärdenfors. *Belief Revision.* Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1992.

[16] Radim Jirousek, Václav Kratochvíl, and Prakash P. Shenoy. On conditional belief functions in the dempster-shafer theory. In Sylvie Le Hégarat-Mascle, Isabelle Bloch, and Emanuel Aldea, editors, *Belief Functions: Theory and Applications - 7th International Conference, BELIEF 2022, Paris, France, October 26-28, 2022, Proceedings*, volume 13506 of *Lecture Notes in Computer Science*, pages 207–218. Springer, 2022.

[17] Jérôme Lang. *Possibilistic Logic: Complexity and Algorithms*, pages 179–220. Springer Netherlands, Dordrecht, 2000.

[18] Ronald Meester and Timber Kerkvliet. A new look at conditional probability with belief functions. *Statistica neerlandica*, 73(2):274–291, 2019.

[19] Lalintha G. Polpitiya, Kamal Premaratne, Manohar N. Murthi, Stephen J. Murrell, and Dilip Sarkar. Efficient computation of conditionals in the dempster-shafer belief theoretic framework. *IEEE Trans. Cybern.*, 52(5):2931–2941, 2022.

[20] Nicolas Schwind, Sébastien Konieczny, Jean-Marie Lagniez, and Pierre Marquis. On computational

aspects of iterated belief change. In Christian Bessiere, editor, *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020*, pages 1770–1776. ijcai.org, 2020.

[21] Glenn Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[22] Wolfgang Spohn. Ordinal conditional functions: A dynamic theory of epistemic states. In *Causation in decision, belief change, and statistics: Proceedings of the Irvine Conference on Probability and Causation*, pages 105–134. Springer, 1988.

[23] Wolfgang Spohn. *The Laws of Belief - Ranking Theory and Its Philosophical Applications*. Oxford University Press, 2014.

[24] Lotfi Zadeh. Fuzzy logic—a personal perspective. *Fuzzy Sets and Systems*, 281, 05 2015.