Topological completeness of extensions of **S4**

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1 Introduction

Perhaps the most celebrated topological completeness result in modal logic is the McKinsey-Tarski theorem that if we interpret modal diamond as topological closure, then **S4** is complete for the real line or indeed any dense-in-itself separable metrizable space [10]. This result was proved before relational semantics for modal logic was introduced. In the last 15 years, utilizing relational semantics for **S4**, a number of different proofs of this result appeared in the literature. Completeness of **S4** for the real line can be found in [1, 4, 13, 12], for the rational line in [2, 12], and for the Cantor space in [11, 1].

For a topological space X, let X^+ be the closure algebra of all subsets of X. Then completeness of **S4** for the real line $\mathbf{R}$ means that **S4** is the modal logic of the closure algebra $\mathbf{R}^+$, and the same is true for the rational line $\mathbf{Q}$ and the Cantor space $\mathbf{C}$.

In [3], the notion of a connected normal extension of **S4** was introduced, and it was shown that each connected normal extension of **S4** that has the finite model property (FMP) is the modal logic of a subalgebra of $\mathbf{R}^+$. It was also shown that each normal extension of **S4** that has FMP is the modal logic of a subalgebra of $\mathbf{Q}^+$, as well as the modal logic of a subalgebra of $\mathbf{C}^+$. It was left as an open problem [3, p. 306, Open Problem 2] whether a connected normal extension of **S4** without FMP is also the modal logic of some subalgebra of $\mathbf{R}^+$.

Our purpose here is to solve this problem affirmatively by showing that each connected normal extension of **S4** (with or without FMP) is in fact the modal logic of some subalgebra of $\mathbf{R}^+$. We also prove that each normal extension of **S4** (with or without FMP) is the modal logic of a subalgebra of $\mathbf{Q}^+$, as well as the modal logic of a subalgebra of $\mathbf{C}^+$. These results generalize similar results from [3] for normal extensions of **S4** with FMP to all normal extensions of **S4**.

2 Closure algebras

We recall [10] that a *closure algebra* is a pair $\mathfrak{A} = (B, \mathbf{C})$, where B is a Boolean algebra and $\mathbf{C}$ is a unary function on B satisfying Kuratowski's axioms: (i) $a \le \mathbf{C}a$, (ii) $\mathbf{CC}a \le \mathbf{C}a$, (iii) $\mathbf{C}(a \vee b) = \mathbf{C}a \vee \mathbf{C}b$, and (iv) $\mathbf{C}0 = 0$. We refer to $\mathbf{C}$ as a *closure operator* on B. Its dual *interior operator* is given by $\mathbf{I}a = -\mathbf{C} - a$, where $-$ is Boolean complement. Closure algebras are also known as interior algebras [6], topological Boolean algebras [14], or **S4**-algebras [7]. The last name is suggestive in that closure algebras are algebraic models of **S4** (see, e.g., [14, 7]).

We call an element a of a closure algebra $\mathfrak{A}$ *closed* if $a = \mathbf{C}a$, *open* if $a = \mathbf{I}a$, and *clopen* if it is both closed and open. A closure algebra $\mathfrak{A}$ is *connected* if 0 and 1 are the only clopen elements of $\mathfrak{A}$, and it is *well-connected* if for closed elements c, d of $\mathfrak{A}$, from $c \wedge d = 0$ it follows that $c = 0$ or $d = 0$ (equivalently, for open elements u, v of $\mathfrak{A}$, from $u \vee v = 1$ it follows that $u = 1$ or $v = 1$). Clearly each well-connected closure algebra is connected, but not conversely.

Each closure algebra $\mathfrak{A}$ can be represented as a subalgebra of the closure algebra X^+ for some topological space X [10]. Moreover, $\mathfrak{A}$ is connected iff X is a connected space [3].

Let L be a normal extension of **S4**. Following [3, Def. 4.1], we call L *connected* if L is the modal logic of some connected closure algebra $\mathfrak{A}$. In other words, if we denote the modal logic of a closure algebra $\mathfrak{A}$ by $L(\mathfrak{A})$, then L is connected iff there exists a connected closure algebra $\mathfrak{A}$ such that $L = L(\mathfrak{A})$. One of the main results of [3] is that if L is a normal extension of **S4** that has FMP, then L is connected iff $L = L(\mathfrak{A})$ for some subalgebra $\mathfrak{A}$ of $\mathbf{R}^+$. It is also shown in [3] that for each normal extension L of **S4** with FMP, there is a subalgebra $\mathfrak{B}$ of $\mathbf{Q}^+$ such that $L = L(\mathfrak{B})$, as well as a subalgebra $\mathfrak{C}$ of $\mathbf{C}^+$ such that $L = L(\mathfrak{C})$. Below we describe how to generalize these results to all normal extensions of **S4** (with or without FMP).

3 Countable general frame property and completeness for Q

We recall that a normal modal logic L has the *finite model property* (FMP) if for each non-theorem φ of L, there is a finite frame $\mathfrak{F}$ validating L and refuting φ, and that L has the *countable model property* (CMP) if such a frame is countable. It is well known that there exist normal modal logics (in particular, normal extensions of **S4**) that have neither FMP nor CMP. Nevertheless, we show that each normal modal logic has what we call the *countable general frame property.*

For a frame $\mathfrak{F} = (W, R)$, let $\mathfrak{F}^+$ be the modal algebra of all subsets of $\mathfrak{F}$. We recall that a *general frame* is a triple $\mathfrak{F} = (W, R, P)$, where (W, R) is a frame and P is a subalgebra of $(W, R)^+$.

Definition 3.1. *Let L be a normal modal logic. We say that L has the* countable general frame property (CGFP) *if for each non-theorem φ of L, there is a countable general frame $\mathfrak{F}$ validating L and refuting φ.*

Theorem 3.2. *Each normal modal logic L has* CGFP*.*

We recall that a frame $\mathfrak{F} = (W, R)$ is an **S4**-*frame* if R is reflexive and transitive. A subset U of W is an R-*upset* if wRu and $w \in U$ imply $u \in U$. The collection τ_R of all R-upsets forms a topology on W, called an *Alexandroff topology*, in which each point has a least neighborhood. The least neighborhood of $w \in W$ is $R[w] = \{u \in W : wRu\}$ (the R-upset *generated* by w). We view **S4**-frames as Alexandroff topological spaces.

For topological spaces X, Y, we recall that a map $f : X \to Y$ is *interior* if it is continuous (the inverse image of every open is open) and open (the direct image of every open is open). It is a consequence of [5, Lem. 3.1] that each countable rooted **S4**-frame is an interior image of the rational line $\mathbf{Q}$. Now, let L be a normal extension of **S4**. By Theorem 3.2, each non-theorem of L is refuted on a countable general frame $\mathfrak{F} = (W, R, P)$ of L, and we can assume that $\mathfrak{F}$ is rooted. Then $\mathfrak{F}$ is an interior image of $\mathbf{Q}$, giving that P is isomorphic to a subalgebra of $\mathbf{Q}^+$. Since we can enumerate non-theorems of L and a countable disjoint union of $\mathbf{Q}$ is homeomorphic to $\mathbf{Q}$, we obtain that there is a subalgebra $\mathfrak{A}$ of $\mathbf{Q}^+$ such that $L = L(\mathfrak{A})$. Thus, we arrive at the following theorem.

Theorem 3.3. *For every normal extension L of* **S4** *there is a subalgebra $\mathfrak{A}$ of $\mathbf{Q}^+$ such that $L = L(\mathfrak{A})$.*

4 Well-connected logics and completeness for $\mathfrak{T}_2$

Let $\mathfrak{T}_2 = (T_2, \leq)$ be the infinite binary tree. As with $\mathbf{Q}$, we have that each countable rooted **S4**-frame $\mathfrak{F} = (W, R)$ is an interior image of $\mathfrak{T}_2$. In fact, since both $\mathfrak{T}_2$ and $\mathfrak{F}$ are **S4**-frames, in this context an interior map simply means a p-morphism, so $\mathfrak{F}$ is a p-morphic image of $\mathfrak{T}_2$. Therefore, for a normal extension L of **S4**, Theorem 3.2 allows one to refute each non-theorem of L on a subalgebra $\mathfrak{A}$ of $\mathfrak{T}_2^+$ that validates L. However, we don't obtain a direct analogue of Theorem 3.3 because a countable disjoint union of $\mathfrak{T}_2$ is not isomorphic to $\mathfrak{T}_2$. Nevertheless, we can prove that a countable disjoint union of $\mathfrak{T}_2$ is isomorphic to a generated subframe of $\mathfrak{T}_2$. Since generated subframes give rise to homomorphic images, we arrive at the following theorem.

Theorem 4.1. *For every normal extension L of* **S4** *there is a subalgebra $\mathfrak{A}$ of a homomorphic image of $\mathfrak{T}_2^+$ such that $L = L(\mathfrak{A})$.*

In general, homomorphic images cannot be dropped from the theorem. Indeed, since $\mathfrak{T}_2$ is rooted, $\mathfrak{T}_2^+$ is well-connected. It is easy to see that a subalgebra of a well-connected algebra is well-connected. As each well-connected algebra is connected, we see that if $L = L(\mathfrak{A})$ for some subalgebra $\mathfrak{A}$ of $\mathfrak{T}_2$, then L is connected. Since not every normal extension of **S4** is connected [3], it follows that there exist normal extensions of **S4** that are not of the form $L(\mathfrak{A})$, where $\mathfrak{A}$ is a subalgebra of $\mathfrak{T}_2^+$.

Definition 4.2. *Let L be a normal extension of* **S4**. *We call L* well-connected *if $L = L(\mathfrak{A})$ for some well-connected closure algebra $\mathfrak{A}$.*

Theorem 4.3. *A normal extension L of* **S4** *is well-connected iff $L = L(\mathfrak{A})$ for some subalgebra $\mathfrak{A}$ of $\mathfrak{T}_2^+$.*

5 The infinite binary tree with limits and completeness for C and R

Although $\mathfrak{T}_2$ is an interior image of $\mathbf{Q}$ [2], $\mathfrak{T}_2$ is neither an interior image of $\mathbf{R}$ nor of $\mathbf{C}$. Because of this we add to $\mathfrak{T}_2$ "leaves," which can be realized as limit points of $\mathfrak{T}_2$ via multiple topologies. We call the resulting space the *infinite binary tree with limits* and denote it by $\mathfrak{L}_2 = (L_2, \leq)$. This uncountable tree has been an object of recent interest [9, 8]. In particular, [8] uses $\mathfrak{L}_2$ in a crucial way to obtain strong completeness of **S4** for any dense-in-itself metric space.

We view T_2 as a subset of L_2. For $t \in T_2$, let ${\uparrow}t = \{s \in L_2 : t \leq s\}$. Let also $\mathfrak{R} = \{{\uparrow}t : t \in T_2\}$ and $\mathfrak{B}(\mathfrak{R})$ be the Boolean subalgebra of the powerset of L_2 generated by $\mathfrak{R}$. We let τ be the topology on L_2 generated by the basis $\mathfrak{R}$ and π be the topology generated by the basis $\mathfrak{B}(\mathfrak{R})$.

Lemma 5.1.

1. *τ is the Scott topology of the dcpo $(L_2, \leq)$ and (L_2, τ) is a spectral space.*

2. *π is the patch topology of τ, $\leq$ is the specialization order of (L_2, τ), and $(L_2, \leq, \pi)$ is the Priestley space corresponding to the spectral space (L_2, τ).*

3. ***C** is homeomorphic to $L_2 - T_2$ and (L_2, π) is the Pelczynski compactification of the discrete space T_2.*

By [8], $\mathfrak{T}_2^+$ is isomorphic to a subalgebra of $\mathfrak{L}_2^+$ (where $\mathfrak{L}_2^+$ is the closure algebra of $\mathfrak{L}_2$ with the Scott topology). This together with Theorem 4.3 yields.

Theorem 5.2. *A normal extension L of* **S4** *is well-connected iff $L = L(\mathfrak{A})$ for some subalgebra $\mathfrak{A}$ of $\mathfrak{L}_2^+$.*

A key advantage of $\mathfrak{L}_2$ over $\mathfrak{T}_2$ is that (L_2, τ) is an interior image of both **R** and **C**. From this, generalizing the technique of [3], we obtain.

Theorem 5.3. *For every normal extension L of* **S4** *there is a subalgebra $\mathfrak{A}$ of $\mathbf{C}^+$ such that $L = L(\mathfrak{A})$.*

Theorem 5.4. *A normal extension L of* **S4** *is connected iff $L = L(\mathfrak{A})$ for some subalgebra $\mathfrak{A}$ of $\mathbf{R}^+$.*

Theorem 5.4 solves [3, p. 306, Open Problem 2].

References

[1] M. Aiello, J. van Benthem, and G. Bezhanishvili, *Reasoning about space: The modal way*, J. Logic Comput. **13** (2003), no. 6, 889–920.

[2] J. van Benthem, G. Bezhanishvili, B. ten Cate, and D. Sarenac, *Multimodal logics of products of topologies*, Studia Logica **84** (2006), no. 3, 369–392.

[3] G. Bezhanishvili and D. Gabelaia, *Connected modal logics*, Arch. Math. Logic **50** (2011), no. 3-4, 287–317.

[4] G. Bezhanishvili and M. Gehrke, *Completeness of S4 with respect to the real line: revisited*, Ann. Pure Appl. Logic **131** (2005), no. 1-3, 287–301.

[5] G. Bezhanishvili and J. Lucero-Bryan, *More on d-logics of subspaces of the rational numbers*, Notre Dame J. Form. Log. **53** (2012), no. 3, 319–345.

[6] W. Blok, *Varieties of interior algebras*, Ph.D. thesis, University of Amsterdam, 1976.

[7] A. Chagrov and M. Zakharyaschev, *Modal logic*, Oxford Logic Guides, vol. 35, The Clarendon Press, New York, 1997.

[8] Ph. Kremer, *Strong completeness of S4 for any dense-in-itself metric space*, Review of Symbolic Logic (2013), forthcoming.

[9] T. Lando, *Completeness of S4 for the Lebesgue measure algebra*, J. Philos. Logic **41** (2012), no. 2, 287–316.

[10] J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Annals of Mathematics **45** (1944), 141–191.

[11] G. Mints, *A completeness proof for propositional* S4 *in Cantor space*, Logic at work, Stud. Fuzziness Soft Comput., vol. 24, Physica, Heidelberg, 1999, pp. 79–88.

[12] ———, *S4 is topologically complete for* $(0, 1)$*: a short proof*, Log. J. IGPL **14** (2006), no. 1, 63–71.

[13] G. Mints and T. Zhang, *A proof of topological completeness for S4 in* $(0, 1)$, Ann. Pure Appl. Logic **133** (2005), no. 1-3, 231–245.

[14] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Monografie Matematyczne, Tom 41, Państwowe Wydawnictwo Naukowe, Warsaw, 1963.