The w(p) in the financial markets: An empirical approach on the S&P 500

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June 26, 2020

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Abstract. The aim of the work is to estimate the probability weighting function, starting from the time series of the S&P 500 index. After an introduction to the Efficient Markets Hypothesis (EMH) and the empirical evidence against it, we have introduced the Prospect Theory (PT). Following the studies carried out by Gonzalez et al., we have analyzed w(p) and we have proposed a new estimation method with a two-parameters function. The OLS (Ordinary Least Squares) method provides the alpha and beta coefficients, which represent respectively the curvature and elevation of the weighting function. In the last part of the paper, w(p) has been implemented in the building of the portfolio with random weights.

Keywords: Probability Weighting Function, Efficient Market Hypothesis, Behavioral Finance.

1 Introduction

1.1 Efficient Market Hypothesis

The concept of efficient market is related to the one of randomness. In 1900 Louis Bachelier in his PhD thesis stated that the prices of market’s stocks followed a random behaviour and it was impossible to forecast their trend through time. From a mathematical point of view it can be affirmed that the price stocks’ variations, in discrete time, move as a random walk characterized from iid (independent and identically distributed) random variations.
At the beginning of 60’s Fama adopted a mathematical/statistical approach to the study of financial markets. He identified three forms of informative efficiency:

- weak form efficiency: prices observed on the market reflect the information contained in their historical series

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[1]probability weighting function.

- semi-strong efficiency: not only do prices reflect the information contained in time series, but any other public information as well.
- strong efficiency: market's prices reflect, beyond what previously said, any private information; it is not possible to set up a trading strategy with an expected efficiency superior to the one of the market basing on any private/favorite information.

In his work Fama endorsed the fact that an efficient market behaved like a martingale belt, in other words, the expected value conditioned by the informative set (filtration) was equal to the price of the stock at the time t, $E(S_{t+1}|F_t)= S_t$.

The miscalculation in the forecast can be defined as:

$$\varepsilon_{t+1} = S_{t+1} - E_t(S_{t+1}),\text{with } E(\varepsilon_t) = 0 \text{ and } cov(\varepsilon_t;\ F_{t-1})=0$$

The last property is known as *orthogonality* and it indicates that the term of error is due to only unforecastable shocks of the market. From a mathematical point of view, a useful instrument to model the wandering behavior of the market is the Brownian motion. It belongs to the processes of Levy, stochastic processes with stationary and independent increments.

Brownian motion has certain characteristics that make him compatible with the Efficient of markets' hypothesis:

- independent and stationary increments;
- no differentiability;
- martingale;
- the increments are: $W(t_2)$-$W(t_1)$ ~$N(0, t_2$-$t_1)$.

Geometric Brownian motion is introduced to ensure the non-negativity of the assets' prices. As matter of fact:

$$St= S_0 * \exp(\mu t+\sigma W(t));\ \log[S(t)]= S_0+\log(\mu t+\sigma W(t)); \qquad (2)$$

$$\log(\mu t+\sigma W(t))=\log S(t)-\log S(0)=\log\frac{S_t}{S_0} \qquad (3)$$

Where:
S_0 is the asset's price at t=0, μ is the drift, σ is the diffusion coefficient, W(t) is the Brownian motion (Bm).

Due to the "stylized facts", see (Cont, Bianchi) in the financial markets, the literature introduced a new method to model St. The new approach is the "fractional Brownian motion" with Hurst exponent (0≤H≤1).

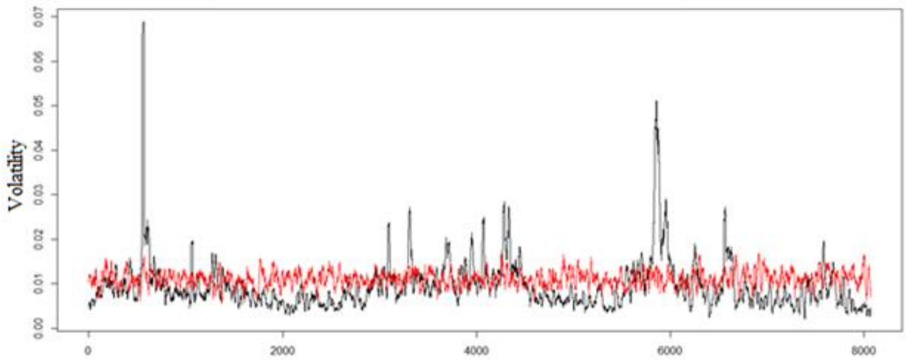


Fig. 1. Differences between empirical and theoretical volatility

The black line is the empirical volatility (computes on the index SP500) while the red line is the volatility that we have estimated from a sample of random variables IID (returns) extracted from a Gaussian distribution that has the same mean value and the same variance of the SP500. The empirical observations are not independent variables, in fact the run test shows:

Table 1. Run Test

Run Test	P-value
Empirical series	0.0025
Simulated series	0.75

The figure 1 underlines the fractionality of the Brownian motion. *If the independence hypothesis of the observations is released, the volatility loses stationarity.*

1.2 Critics to the efficient markets and introduction to Behavioral Finance

In an efficient market the price of an asset is evaluated from the current value of the expected dividends with a rate corrected by the risk (DDM). Many empirical researches demonstrated the quotations are far from the equilibrium (theoretical) and they fix just in the long period.
During 50's Herbert Simon was talking of partially rational economical agents: "It is not empirically evident that entrepreneurs and consumers follow the benefit maximization principles required by marginalist models when taking a decision". Many empirical researches have been conducted in time to verify the strength of theoretical hypothesis, especially the first two forms of efficiency. Keown and Pinkerton tested the hypothesis of semi strong efficiency (event study), observing what happens when news about the acquisition and merging of companies diffuse. Assets' price does not grow steadily, it begins with a slow increase in the days before the announcement and the not reaches the maximum in the day when the news is published. Market's returns and other indicators, as P/E ratio, show a high volatility not justifiable by the fundamentals of companies and the arrival of new information. In the real world, the resolution of complex

individual choices is made through mental rules known as: heuristics. Given the constraints, in which the individuals operate, the heuristics determine a solution (the solution can be different from the optimum choice).
Kahneman and Tversky have shown that individuals in the choices of information are influenced by the ease with which they can be recalled in the mind. The authors argue that when an individual faces his mental processes, he will unbalance probability judgment in favor of easier cases that can represented and he distort chances even when it is easier to recall family situations.
Representative heuristics foresees that the operator is based on stereotypes or on familiar situations for the formulation of probability judgments. When the realization of a random phenomenon is considered representative, sometimes it can produce paradoxical consequences.

1.3 The Prospect Theory

In the Prospect Theory, as we previously observed, individuals do not use linearly the probability (unlike the Expected Utility Theory or EUT). EUT is a special case in which w(p)=p, the model is:

$$v(U,p)=\sum_{i=1}^{n} p_i * U(x_i);\ \text{where: } U(x_i) \in R^{+} \quad (4)$$

[2]

In the Prospect Theory, w(p) acts as a subjective filter putting emphasis upon two aspects:

- the lowest probabilities are overestimated;
- high probabilities are underestimated.

The classic version is:

$$v(x,p)=\sum_{i=1}^{n} w(p_i) * V(x_i); \quad (5)$$

where:

- $w(p_i)$ is the weighting function;
- $V(x_i)$ is the value function.

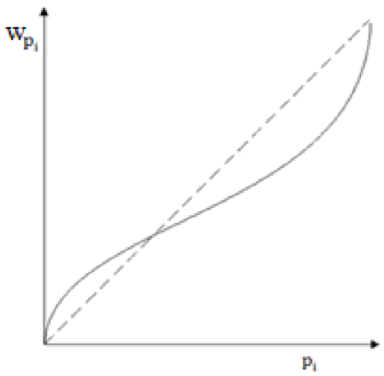
$w(p_i)$ is a function that:

- overweight low probability;
- underweight high probabilities.

While, $V(x_i)$ is:

[2] The biggest difference with the value function $V(x_i)$.

$$V(x)=\begin{cases} x^{\alpha} with\ x > 0 \\ -\beta(-x^{\alpha})\ \text{with}\ x < 0 \end{cases};\ \text{where } \beta \text{ is the loss aversion coefficient.} \quad (6)$$



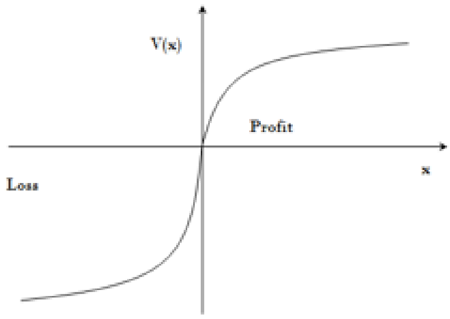


Fig. 2. Probability weighting function **Fig. 3.** Value function

The function $\pi(p)$ was introduced late by Kahneman end Tversky, in fact the first version of the PT $\sum_{i=1}^{n} w_{(p_i)} * V(x_i)$ violated the stochastic dominance, first order, (see Quigging).

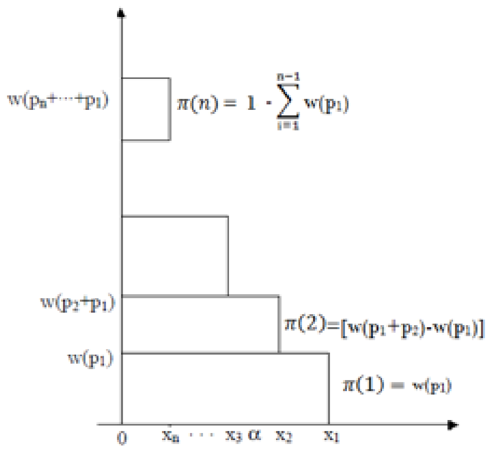


Fig. 4. Introduction of rank

A variant of PT is defined by Rieger and Wang (2008), the function form of v(x,p) is:

$$v(x,p)=\frac{\sum_{i=1}^{n} w(p_i)*V(x_i)}{\sum_{i=1}^{n} w(p_i)} \quad (7)$$

2 The Probability Weighting Function w(p)

2.1 The interpretation of w(p)

In the literature, there is specific interest in weighting functions that are initially concave, related to low probabilities in an interval (0, δ), for 0 < δ < 1, and convex for medium and large probabilities, on (δ, 1). We call these functions inverse - S shaped weighting functions, reflecting the shape of the corresponding mapping. Related to the curvature of weighting functions is the notion of probabilistic risk aversion. A convex weighting function characterizes probabilistic risk aversion (or pessimism) whereas a concave weighting function characterizes risk proneness (or optimism). A linear weighting function is characterized by probabilistic risk neutrality.
The concept of diminishing sensitivity is linked to the notion of discriminability The function w1 shows greater discriminability than w_2 when:

$$w_1(p+\varepsilon) - w_1(p) > w_2(p+\varepsilon) - w_2(p) \quad (8)$$

with p ϵ [0,1], and ε ϵ R^+.

This means that changes that occur in w_1 are bigger than changes w_2.
The notion, which Tversky and Kahneman called diminishing sensitivity, is very simple: people become less sensitive to changes in probability as they move away from a reference point. It stated that when people step further away from a reference point, they tend to become less sensitive to changes in probability. In the odds domain, the two endpoints 0 and 1 serve as reference points in the sense that 0 represents “certainly will not happen” and 1 represents “certainly will happen”.

Under the principle of diminishing sensitivity, increments near the end points of the probability scale loom larger than increments near the middle of the scale. Attractiveness can be defined as a person finds chance domain 1 more attractive than chance domain 2 if $w_1(p) > w_2(p)$ for all p ϵ [0,1] see (Gonzalez). Elevation has also an interesting interpretation as a measure of relative optimism (see Abdellaoui et al.).

2.2 The form of w(p)

Many different parametric functional forms have been proposed in the literature. In this section, we analyze some families of w(p) in which a single parameter determines the nature and the magnitude of the discrepancy between the transformed probabilities, w(p), and the original ones, p, by capturing features such as the curvature and the elevation of the function and the position of the fixed point (w(p) = p). Karmarkar proposed (one parameter function):

$$log\frac{w(p)}{1-w(p)}=\beta\ log\frac{p}{1-p};\text{ with } \beta \in R^{+} \quad (9)$$

In 1979, Kahneman and Tverky used a generalization of one parameter function:

$$w(p)=\frac{p^{\Upsilon}}{(p^{\Upsilon}+(1-p)^{\Upsilon})^{1/\Upsilon}}\text{ ; with } \Upsilon>0.278. \quad (10)$$

The form of w(p) with two parameters

There are two parameters in the w(p) estimation that drive the function form, the curvature and the elevation, in the Gonzalez paper we have the following form:

$$log\frac{w(p)}{1-w(p)}=\alpha+\beta\ log\frac{p}{1-p} \quad (11)$$

where:

- α controls the elevation (intercept);
- β controls the curvature (slope).

This model is linear in the log-odds,

$$\text{in fact } w(p)=\frac{\lambda p^{\beta}}{\lambda p^{\beta}+(1-p)^{\beta}}\text{ ; where } \lambda=\exp(\alpha) \quad (12)$$

Another two-parameter weighting function that also varies curvature and elevation separately was proposed by Prelec (1998). The functional form is:

$$w(0)=0$$

$$w(1)=1$$

$$w(p)=\exp(-\lambda(-\log(p))^{\beta}) \quad (13)$$

see al-Nowaihiy and Dhami (2010) for further information.

Note that a probability weighting function by itself, is not a theory of risk. It needs to be embedded within other theories, such as RDU, PT, for it to have significant predictive content in concrete economic situations. We use w(p) in portfolio model to compute the expected value and the volatility (σ), through the building of portfolio frontiers with random weights.

3 Estimation of w(p)

3.1 The model

The aim of this paper is the estimation of $w(p)$ on the financial markets, in particular on USA stock market, through a statistical approach, generally the function is estimate in an experiment see (Prelec at al.). The phases of this method are:

- the objective probabilities (p) are extracted from a Gaussian distribution N(μ, σ^2). As previously stated (EMH), the price variation follows a Normal distribution or Log-Normal;
- the practical probabilities ($w(p)$) are extracted from an empirical distribution estimated through Kernel Density Estimation;
- integration of distribution for small intervals (Riemann integral);
- find the relationship between empirical and theoretical probabilities.

The financial time series is the S&P500 from 01/01/2016 to 01/01/2017, we calculate the logarithmic returns as: $r_i = \ln(\frac{Ad_{t+1}}{Ad_t})$; where A_d is the adjusted daily price. Furthermore, we estimate the empirical density distribution through the Kernel Density Estimation.
The (KDE) is a non-parametric way to estimate the probability density function of a random variable.
Its kernel density estimator is $f_h(x)=\frac{\sum_{i=1}^{n} K_h(x-x_i)}{n}$; where:

- K is the kernel (a non-negative function that integrates to one);
- h > 0 is a smoothing parameter called the bandwidth. In this case, the optimal choice is $\approx 1.06\sigma n^{-1/5}$.

In the second step, we fit the Gaussian density distribution with μ equal to the logarithm returns average (μ): $\frac{1}{n}\sum_{i=1}^{n} r_i$ and $\sigma = \sqrt{\sum_{i=1}^{n} \frac{(r_i-\mu)^2}{n}}$.

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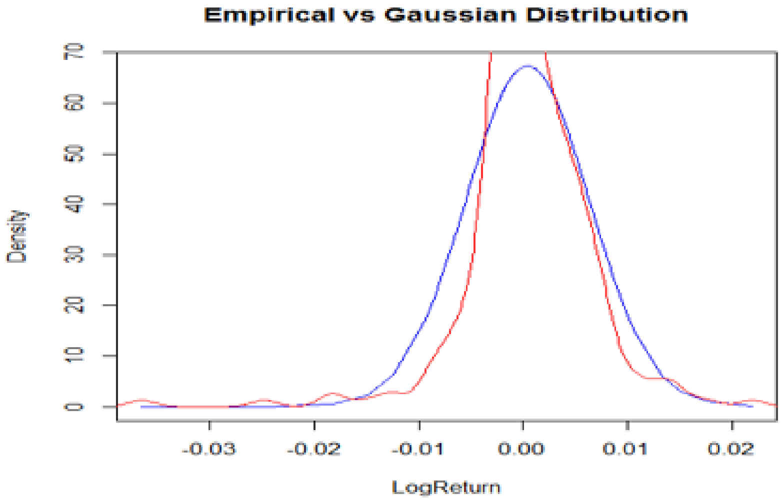

Fig. 5. Empirical density function vs Gaussian

To compute the probabilities, we integrate both the density functions in small intervals, we know that:
$\int_{-\infty}^{+\infty} f(x)dx = 1$ and f(x)>0, where f(x) is a generic density function, $\int_{r_i}^{r_{i+1}} f(x)$, we obtaine a probability range.

$$\text{If } [r_i, r_{i+1}] \rightarrow 0;\ \int_{r_i}^{r_{i+1}} f(x) \approx \text{p} \qquad (14)$$

3.2 Relationship between w(p) and p

We estimate the model through the OLS (ordinary least square). The function (regression curve) should minimize the sum of squares of the distances between the theoretical values and the ones observed.

$$f(\beta_0, \beta_1) = \min \sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2 = \text{Min} \sum_{i=1}^{n} [y_i - \hat{y}_i]^2 \qquad (15)$$

- y_i are observed values;
- $\hat{y}_i$are theoretical values.

We use the “linear in log odds”, the OLS is a valid method because the linearity is on the coefficients not on the variables. The model is:

$$log \frac{w(p)}{1-w(p)} = \alpha + \beta \; log \frac{p}{1-p} + \epsilon \qquad (16)$$

- α is the intercpet;
- $beta$ is the slope;

- ϵ is a gaussian WN~ $(0,\sigma^2)$.

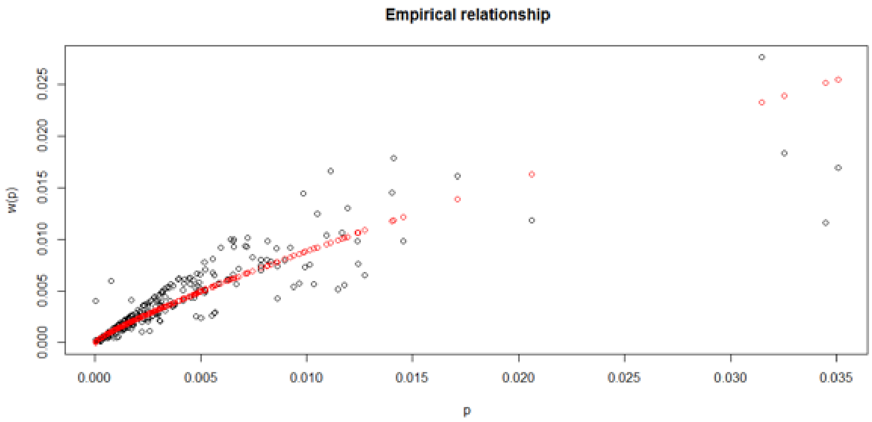

Fig. 6. Relationship between w(p) and p

The linearity is:

$$E(log\,\frac{w(p)}{1-w(p)}\,|log\,\frac{p}{1-p})=\alpha+\beta\;log\,\frac{p}{1-p} \qquad (17)$$

where α =-0.86; β=0.84

1. w (0) =0
2. w (1) = 1
3. R^2= 0.84 shows a good fit of the model.

$$w(p)=\frac{\lambda p^{\beta}}{\lambda p^{\beta}+(1-p)^{\beta}};\; w(p)=\frac{0.42*p^{0.84}}{0.42p^{0.84}+(1-p)^{0.84}};\; \lambda=\exp(\alpha) \qquad (18)$$

The figure 8 shows a good fit of the model to the data. The green line is the empirical relationship, while the red line is the theoretical values.

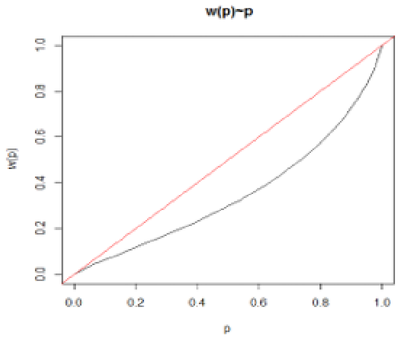

Fig. 7. The w(p) on the SP&500

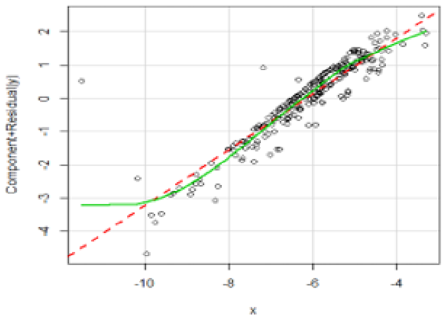

Fig. 8. Fitting regression

Following Gonzales paper, we show that the parameters β and λ control influence the curvature (discriminability) and the elevation (attractiveness). The figure 9 shows the w(p) function for 0.2 ≤β≤ 1.8 and fixed λ (0.43), while fig.10 displays the w(p) function for fixed β (0.84) and 0.2≤λ≤1.8. Note that the red line is the w(p) with OLS parameters.

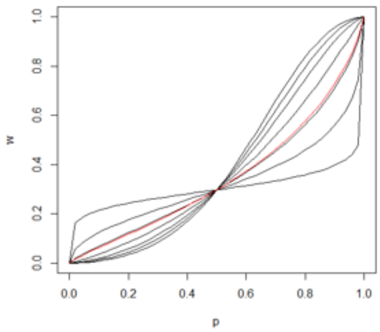

Fig. 9. Changes in curvature

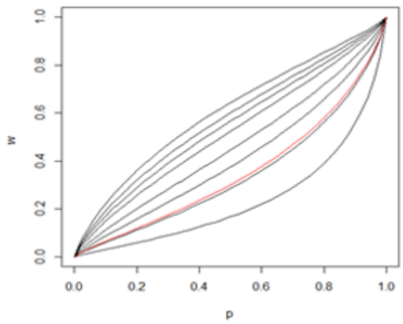

Fig. 10. Changes in elevation

3.3 Financial Application

In order to incorporate w(p) in a financial theory, we use the portfolio theory. According to Markowitz and supposed that the probability law of the S_t is recognized, we build two portfolios with random weights: x_i (uniformly distributed random variable) and (1-x_i).

The first frontier is given by:

$$\mu_p = x_i * \mu_a + (1 - x_i) * \mu_b \tag{19}$$

where:

$$\mu = \sum_{i=1}^{n} p_i * r_i \text{ and } \sigma^2 = \sum_{i=1}^{n} (r_i - \mu)^2 * p_i \tag{20}$$

$$\sigma_p^2 = {x_i}^2 * \sigma_a^2 + (1 - x_i)^2 * \sigma_b^2 + 2 * x_i * (1 - x_i) * \sigma_{r_a r_b} \tag{21}$$

where:

- σ_a^2 and σ_b^2 are the variance of assets a and b;
- $\sigma_{r_a r_b}$ is the covariance.

The second frontier is given by:

$$\mu_{w(p)} = x_i * \mu_c + (1 - x_i) * \mu_d \tag{22}$$

where:

$$\mu = \sum_{i=1}^{n} w(p_i) * r_i \tag{23}$$

and

$$\sigma^2 = \sum_{i=1}^{n} (r_i - \mu)^2 * w(p_i) \tag{24}$$

$$\sigma_w^2 = {x_i}^2 * \sigma_c^2 + (1 - x_i)^2 * \sigma_d^2 + 2 * x_i * (1 - x_i) * \sigma_{r_c r_d} \tag{25}$$

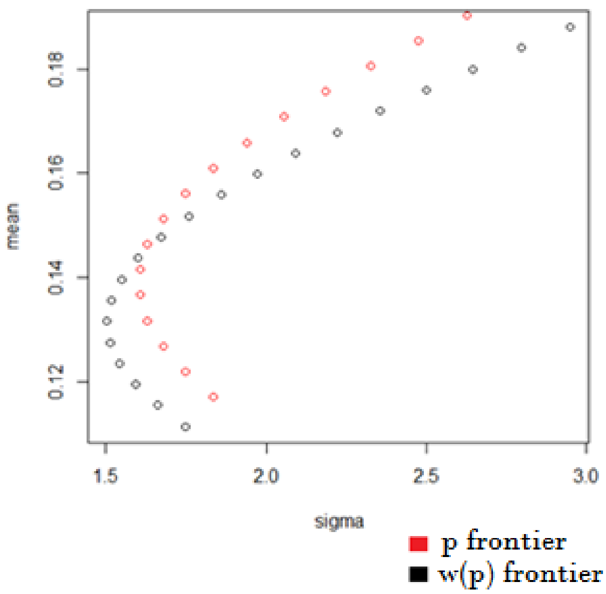

Fig.11. Portfolio Frontiers

Further application

The probability weighting function w(p) can be applied in other financial models in particular it could be used to estimate the views vector (Q) in the Black and Litterman model.

References

1. Abdellaou, M., Parameter-free elicitation of utility and probability weighting functions, Management science, 1497-1512 (2000).
2. Al-Nowaihi A., Dhami S., Probability weighting functions, Wiley Encyclopedia of Operations Research and Management Science, 7-16 (2010).
3. Bachelier L., Louis Bachelier's theory of speculation: the origins of modern finance, Princeton University Press, 6-11 (2011).
4. Benartzi S., Thaler R., Heuristics and Biases in Retirement Savings Behavior, The journal of economic perspectives, 81-104 (2001).
5. Bianchi S., Pianese A., Multifracional Process in Finance, SSRN, 2-6 (2013).
6. Bruce I. J., Kenneth N. L., Calendar Anomalies: Abnormal Returns at Calendar Turning Points,Financial Analysts Journal, Vol. 44, No. 5, 78-88 (1988).

7. Carena A., Mastrogiorgio A., La trappola del comandante: Alla scoperta degli errori cognitivi che ci impediscono di decidere correttamente, Etas, Firenze (2012).
8. Cont R., Empirical properties of asset returns: stylized facts and statistical issues, Taylor, 224 -227 (2001).
9. Di Sciorio F., Dalla Modern Portfolio Theory alla Behavioral Finance: il passaggio evolutivo, Research Gate, 8-25 (2016).
10. Fama E., The behavior of stock market prices, Journal of Business, 35-60 (1965).
11. Farrell Jr., James L., The dividend discount model: A primer, Financial Analysts Journal,16-25 (1985).
12. Goldberger A. S., Best linear unbiased prediction in the generalized linear regression model, Journal of the American Statistical Association, 369-375 (1962).
13. Gonzalez R., Wu G., On the shape of the probability weighting function, Cognitive psychology, 129-166 (1999).
14. Humphrey S.J., Do individuals learn not to make irrational choices, Working paper. Centre for Decision Research and Experimental Economics, School of Economics, 1-4 (2001).
15. Jacobsen B., Bouman S., The Halloween Indicator, Sell in May and Go Away: Another Puzzle, American Economic Review, Vol. 92, No. 5, 1618-1635 (2002).
16. Kahneman D., Tversky A., Prospect Theory: An Analysis of Decision under Risk, Econometrica, 278-279 (1979).
17. Markowitz H. M., Portfolio Selection, Journal of Finance, 77-91 (1952).
18. Markowitz H. M., Mean—Variance Analysis, Finance, Spring, 194-198 (1989).
19. Mosteller F., Tukey J. W., Data analysis and regression: a second course in statistics, Addison-Wesley Series in Behavioral Science: Quantitative Methods, (1977).
20. Nardon M., Pianca P., Probability weighting functions, SSRN, 5-15 (2015).
21. Prelec D., The probability weighting function, Econometrica, 497-527 (1998).
22. Quiggin J., A theory of anticipated utility, Journal of Economic Behavior, 323-343 (1982).
23. Rabin M., Risk aversion and expected-utility theory: A calibration theorem, Handbook of the Fundamentals of Financial Decision Making: Part I, 241-252 (2013).
24. Wakker P.P., Prospect theory: For risk and ambiguity, Cambridge university press., Cambridge (2010).