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Jose Arnaldo Bebita Dris and Immanuel Tobias San Diego

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Some Modular Considerations Regarding Odd Perfect Numbers

Jose Arnaldo Bebita Dris[1] **and Immanuel Tobias San Diego**[2]

[1] M. Sc. Graduate, Mathematics Department
De La Salle University, Manila, Philippines 1004
e-mail: josearnaldobdris@gmail.com

[2] Department of Mathematics and Physical Sciences
Trinity University of Asia, Quezon City, Philippines 1102
e-mail: itsandiego@tua.edu.ph

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Abstract: Let $p^k m^2$ be an odd perfect number with special prime p. In this article, we provide an alternative proof for the biconditional that $\sigma(m^2) \equiv 1 \pmod 4$ holds if and only if $p \equiv k \pmod 8$. We then give an application of this result to the case when $\sigma(m^2)/p^k$ is a square.
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1 Introduction

Let $\sigma(z)$ denote the sum of the divisors of $z \in \mathbb{N}$, the set of positive integers. Denote the deficiency [5] of z by $D(z) = 2z - \sigma(z)$, and the sum of the aliquot divisors [6] of z by $s(z) = \sigma(z) - z$. Note that we have the identity $D(z) + s(z) = z$.

If n is odd and $\sigma(n) = 2n$, then n is said to be an odd perfect number [8]. Euler proved that an odd perfect number, if one exists, must have the form $n = p^k m^2$, where p is the special prime satisfying $p \equiv k \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$.

Chen and Luo [2] gave a characterization of the forms of odd perfect numbers $n = p^k m^2$ such that $p \equiv k \pmod 8$. Starni [7] proved that there is no odd perfect number decomposable into primes all of the type $\equiv 1 \pmod 4$ if $n = p^k m^2$ and $p \not\equiv k \pmod 8$. Starni used a congruence from Ewell [3] to prove this result.

Note that, in general, since m^2 is a square, we get

$$\sigma(m^2) \equiv 1 \pmod 2.$$

This paper provides an alternative proof for Theorem 3.3, equation 3.1 in Chen and Luo's article titled "Odd multiperfect numbers" [2]:

Theorem 1.1. *Let $n = \pi^{\alpha} M^2$ be an odd 2-perfect number, with π prime,* $\gcd(\pi, M) = 1$ *and* $\pi \equiv \alpha \equiv 1 \pmod 4$*. Then*

$$\sigma(M^2) \equiv 1 \pmod 4 \iff \pi \equiv \alpha \pmod 8.$$

The method presented in this paper may potentially be used to extend the arguments to consider $\sigma(m^2)$ modulo 8.

2 Preliminaries

Starting from the fundamental equality

$$\frac{\sigma(m^2)}{p^k} = \frac{2m^2}{\sigma(p^k)}$$

(which follows from the facts that $\sigma(n) = 2n$, σ is multiplicative, and $\gcd(p^k, \sigma(p^k)) = 1$) one can derive

$$\frac{\sigma(m^2)}{p^k} = \frac{2m^2}{\sigma(p^k)} = \gcd(m^2, \sigma(m^2))$$

so that we ultimately have

$$\frac{D(m^2)}{s(p^k)} = \frac{2m^2 - \sigma(m^2)}{\sigma(p^k) - p^k} = \gcd(m^2, \sigma(m^2))$$

and

$$\frac{s(m^2)}{D(p^k)/2} = \frac{\sigma(m^2) - m^2}{p^k - \frac{\sigma(p^k)}{2}} = \gcd(m^2, \sigma(m^2)),$$

whereby we obtain

$$\frac{D(p^k)D(m^2)}{s(p^k)s(m^2)} = 2.$$

Note that we also have the following equation:

$$\frac{2D(m^2)s(m^2)}{D(p^k)s(p^k)} = \bigg(\gcd(m^2, \sigma(m^2))\bigg)^2. \qquad (*)$$

Lastly, notice that we can easily get

$$\sigma(p^k) \equiv k + 1 \equiv 2 \pmod 4$$

(since $p \equiv k \equiv 1 \pmod 4$) so that it remains to consider the possible equivalence classes for $\sigma(m^2)$ modulo 4. Since $\sigma(m^2)$ is odd, we only need to consider two.

We ask: Which equivalence class of $\sigma(m^2)$ modulo 4 makes Equation $(*)$ untenable?

3 Discussion and Results

We know that the answer to the question we posed in the previous section must somehow depend on the equivalence class of p and k modulo 8, but as we only know that $p \equiv k \equiv 1 \pmod 4$, we need to consider the following cases separately and thereby prove the corresponding results:

Remark 3.1. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p. We claim the truth of the following propositions, which we will need to treat separately later:*

1. *If $p \equiv k \equiv 1 \pmod 8$, then $\sigma(m^2) \equiv 3 \pmod 4$ is impossible.*
2. *If $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, then $\sigma(m^2) \equiv 1 \pmod 4$ is impossible.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $\sigma(m^2) \equiv 1 \pmod 4$ is impossible.*
4. *If $p \equiv k \equiv 5 \pmod 8$, then $\sigma(m^2) \equiv 3 \pmod 4$ is impossible.*

First, we prove the following lemmas:

Lemma 3.2. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $p \equiv 1 \pmod 8$, then $\sigma(p^k) \equiv k+1 \pmod 8$.*
2. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $\sigma(p^k) \equiv 6 \pmod 8$.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 5 \pmod 8$, then $\sigma(p^k) \equiv 2 \pmod 8$.*

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime p. It follows that $p \equiv 1 \pmod 4$.

We consider two cases:

Case 1: $p \equiv 1 \pmod 8$ We obtain

$$\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv 1 + \sum_{i=1}^{k} p^i \equiv 1 + \sum_{i=1}^{k} 1^i \equiv k+1 \pmod 8,$$

as desired.

Case 2: $p \equiv 5 \pmod 8$ We get

$$\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv \sum_{i=0}^{k} 5^i \equiv \begin{cases} 6 \pmod 8, \text{ if } k \equiv 1 \pmod 8 \\ 2 \pmod 8, \text{ if } k \equiv 5 \pmod 8 \end{cases}$$

□

Lemma 3.3. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p.*

1. *If $p \equiv 1 \pmod 8$ and $k \equiv 1 \pmod 8$, then $D(p^k) \equiv 0 \pmod 8$.*
2. *If $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, then $D(p^k) \equiv 4 \pmod 8$.*
3. *If $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, then $D(p^k) \equiv 4 \pmod 8$.*

4. *If* $p \equiv 5 \pmod 8$ *and* $k \equiv 5 \pmod 8$, *then* $D(p^k) \equiv 0 \pmod 8$.

Proof. The proof is trivial and follows directly from Lemma 3.2, using the formula $D(p^k) = 2p^k - \sigma(p^k)$. ☐

Lemma 3.4. *Suppose that* $n = p^k m^2$ *is an odd perfect number with special prime* p.

1. *If* $p \equiv 1 \pmod 8$ *and* $k \equiv 1 \pmod 8$, *then* $s(p^k) \equiv 1 \pmod 8$.
2. *If* $p \equiv 1 \pmod 8$ *and* $k \equiv 5 \pmod 8$, *then* $s(p^k) \equiv 5 \pmod 8$.
3. *If* $p \equiv 5 \pmod 8$ *and* $k \equiv 1 \pmod 8$, *then* $s(p^k) \equiv 1 \pmod 8$.
4. *If* $p \equiv 5 \pmod 8$ *and* $k \equiv 5 \pmod 8$, *then* $s(p^k) \equiv 5 \pmod 8$.

Proof. The proof is trivial and follows directly from Lemma 3.3, using the formula $s(p^k) = p^k - D(p^k)$. ☐

Lemma 3.5. *Suppose that* $n = p^k m^2$ *is an odd perfect number with special prime* p.

1. *If* $\sigma(m^2) \equiv 1 \pmod 4$, *then* $D(m^2) \equiv 1 \pmod 4$.
2. *If* $\sigma(m^2) \equiv 3 \pmod 4$, *then* $D(m^2) \equiv 3 \pmod 4$.

Proof. The proof is trivial and follows directly from the fact that $m^2 \equiv 1 \pmod 4$ (since m is odd), using the underlying assumptions and the formula $D(m^2) = 2m^2 - \sigma(m^2)$. ☐

Lemma 3.6. *Suppose that* $n = p^k m^2$ *is an odd perfect number with special prime* p.

1. *If* $\sigma(m^2) \equiv 1 \pmod 4$, *then* $s(m^2) \equiv 0 \pmod 4$.
2. *If* $\sigma(m^2) \equiv 3 \pmod 4$, *then* $s(m^2) \equiv 2 \pmod 4$.

Proof. The proof is trivial and follows directly from Lemma 3.5, using the formula $s(m^2) = m^2 - D(m^2)$. ☐

We are now ready to prove our main result.

Theorem 3.7. *Suppose that* $n = p^k m^2$ *is an odd perfect number with special prime* p.

1. *If* $p \equiv k \equiv 1 \pmod 8$, *then* $\sigma(m^2) \equiv 3 \pmod 4$ *is impossible.*
2. *If* $p \equiv 1 \pmod 8$ *and* $k \equiv 5 \pmod 8$, *then* $\sigma(m^2) \equiv 1 \pmod 4$ *is impossible.*
3. *If* $p \equiv 5 \pmod 8$ *and* $k \equiv 1 \pmod 8$, *then* $\sigma(m^2) \equiv 1 \pmod 4$ *is impossible.*
4. *If* $p \equiv k \equiv 5 \pmod 8$, *then* $\sigma(m^2) \equiv 3 \pmod 4$ *is impossible.*

Proof. Let $n = p^k m^2$ be an odd perfect number with special prime p.

Notice that the right-hand side of Equation $(*)$

$$\frac{2D(m^2)s(m^2)}{D(p^k)s(p^k)} = \left(\gcd(m^2, \sigma(m^2))\right)^2. \qquad (*)$$

is odd. (Furthermore, it is congruent to 1 modulo 8.)

First, suppose that $p \equiv k \equiv 1 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 3 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 0 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 3 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 1 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_1 + 3)(4b_1 + 2) = (8x_1 + 1)(8c_1)(8d_1 + 1)$$

which does not have any integer solutions.

Next, suppose that $p \equiv 1 \pmod 8$ and $k \equiv 5 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 1 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 4 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 1 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 5 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 0 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_2 + 1)(4b_2) = (8x_2 + 1)(8c_2 + 4)(8d_2 + 5)$$

which does not have any integer solutions.

Now, suppose that $p \equiv 5 \pmod 8$ and $k \equiv 1 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 1 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 4 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 1 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 1 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 0 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_3 + 1)(4b_3) = (8x_3 + 1)(8c_3 + 4)(8d_3 + 1)$$

which does not have any integer solutions.

Finally, suppose that $p \equiv k \equiv 5 \pmod 8$, and assume to the contrary that $\sigma(m^2) \equiv 3 \pmod 4$ holds. By Lemma 3.3, $D(p^k) \equiv 0 \pmod 8$. By Lemma 3.5, $D(m^2) \equiv 3 \pmod 4$. By Lemma 3.4, $s(p^k) \equiv 5 \pmod 8$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod 4$. Thus, from Equation $(*)$ we obtain (symbolically)

$$2(4a_4 + 3)(4b_4 + 2) = (8x_4 + 1)(8c_4)(8d_4 + 5)$$

which does not have any integer solutions.

This concludes the proof. □

Remark 3.8. *To summarize, Theorem 3.7 just states that if $n = p^k m^2$ is an odd perfect number with special prime p, then $\sigma(m^2) \equiv 1 \pmod 4$ holds if and only if $p \equiv k \pmod 8$. Our argument provides an alternative proof for Theorem 3.3, equation 3.1 in [2] (as reproduced above in Theorem 1.1).*

4 An Application

Let $n = p^k m^2$ be an odd perfect number with special prime p, and let $\sigma(m^2)/p^k$ be a square. Since $\sigma(m^2)/p^k$ is odd, it follows that $\sigma(m^2)/p^k \equiv 1 \pmod 4$. But it is known that $p \equiv k \equiv 1 \pmod 4$. In particular, we know that $p^k \equiv 1 \pmod 4$. This implies that $\sigma(m^2) \equiv 1 \pmod 4$, if $\sigma(m^2)/p^k$ is a square. By Theorem 3.7, we know that $p \equiv k \pmod 8$.

Moreover, Broughan, Delbourgo, and Zhou proved in [1] (Lemma 8, page 7) that if $\sigma(m^2)/p^k$ is a square, then $k = 1$ holds.

Thus, under the assumption that $\sigma(m^2)/p^k$ is a square, we have

$$p \equiv k = 1 \pmod 8.$$

This implies that the lowest possible value for the special prime p is 17.

We state this result as our next theorem.

Theorem 4.1. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p. If $\sigma(m^2)/p^k$ is a square, then $p \geq 17$.*

Remark 4.2. *Let $n = p^k m^2$ be an odd perfect number with special prime p.*

Note that if

$$\frac{\sigma(m^2)}{p^k} = \frac{m^2}{\sigma(p^k)/2}$$

is a square, then $k = 1$ and $\sigma(p^k)/2 = (p+1)/2$ is also a square.

The possible values for the special prime satisfying $p < 100$ and $p \equiv 1 \pmod 8$ are 17, 41, 73, 89, and 97.

For each of these values:

$$\frac{p_1+1}{2} = \frac{17+1}{2} = 9 = 3^2.$$

$$\frac{p_2+1}{2} = \frac{41+1}{2} = 21 \text{ which is not a square.}$$

$$\frac{p_3+1}{2} = \frac{73+1}{2} = 37 \text{ which is not a square.}$$

$$\frac{p_4+1}{2} = \frac{89+1}{2} = 45 \text{ which is not a square.}$$

$$\frac{p_5+1}{2} = \frac{97+1}{2} = 49 = 7^2.$$

A quick way to rule out 41, 73 and 89, as remarked by Ochem [4] over at Mathematics StackExchange, is as follows: "If $(p+1)/2$ is an odd square, then $(p+1)/2 \equiv 1 \pmod 8$, so that $p \equiv 1 \pmod{16}$. This rules out 41, 73, and 89."

5 Conclusion

Additional tools are required if we are to push the analysis from $\sigma(m^2)$ modulo 4 to consider $\sigma(m^2)$ modulo 8. The authors have tried to check Equation $(*)$ by considering $m^2 \equiv 1 \pmod 8$, and the various corresponding cases for $\sigma(m^2)$ modulo 8 (which are determined by Theorem 3.7), but so far all their attempts have not resulted in any contradictions.

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