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Output Feedback Reference Tracking for Constrained Linear Systems via Invariant Sets: The Coupled Tanks Process

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Abstract: The design of a practical output feedback controller applied to the constant reference tracking problem for constrained linear discrete-time systems via set-invariance techniques is presented. In this regard, Output-Feedback Controlled-Invariant (OFCI) Polyhedra are used to ensure that state and input constraints are satisfied at all times. A common coupled tanks process that consists of two interconnected water tanks is used. Based on the measured output and on the linearized dynamics of the system, a suitable control sequence is computed via linear programming such that reference tracking for the real system under constraints is achieved. The practical results show that the control objective is fulfilled and also illustrate performance limitations due to the use of a model approximation and the measurement noise.

Keywords: Linear systems, Invariant sets, Output feedback, Reference tracking, Constraints, Coupled tanks

1. INTRODUCTION

Since the development of Lyapunov theory, the set invariance approach has proven to be very useful for constrained linear systems (Blanchini, 1999; Blanchini and Miani, 2015). Usually the system constraints are imposed (or inherent) on state, control, and output variables. A nonempty set in the state space is said to be positively invariant with respect to (w.r.t.) a given system if for any initial state belonging to such a set, the trajectory of the system state vector does not escape from it (Bitsoris, 1988; Dórea and Hennet, 1999).

In general, constraints on state and control variables can be translated into admissible sets in the state space, that is, regions in which the trajectory of the state vector must be maintained. In the vast majority of cases, the initial set of constraints is not positively invariant and, therefore, it is not guaranteed that the constraints are satisfied at all times. However, when a given set of constraints is not positively invariant under a given dynamics, it is possible to construct a controlled invariant set contained in it (Blanchini, 1994; Dórea and Hennet, 1999). In this case, there is a state feedback control such that, for any initial condition in this set, the trajectory of the state vector does not violate the constraints.

It is a fact that seldom the complete measurement of the state variables of a system is available in practical situations, and then only information from the measured ones, which typically constitute the system output, are available. In such a case, it is necessary to use an output feedback approach. In (Dórea, 2009; Almeida and Dórea, 2021) conditions were established to assess whether a given polyhedral set is Output-Feedback Controlled-Invariant (OFCI), and an output feedback structure was studied. OFCI sets are sets in which the state vector trajectory can be confined through an appropriate sequence of control actions, which are taken based only on the measured outputs.

This work deals with the output feedback control applied to constant reference tracking problem in discrete-time linear systems subject to state and control constraints. The system we consider is a Coupled Tanks Process through which the applicability of the controller is illustrated. Based on the available measurement and an OFCI polyhedral set, a suitable control sequence is computed via Linear Programming (LP) which is able to achieve reference tracking under state and control constraints. The approach used here was developed and studied in more detail in (Almeida and Dórea, 2018; Almeida et al., 2023) for constrained known systems and in (Silveira Jr. and Dórea, 2023) for constrained uncertain linear systems. There, one can also find a comparison with other strategies, particularly the Model Predictive Control (MPC). The main contribution of the present work lies on the practical implementation of the constrained controllers, the determination of the limitations due to the use of a linearized mathematical model to base the control actions to be applied to the real system, and the influence of the measurement noise in the performance of the controller.

2. INVARIANT SETS

Consider the linear, time-invariant, discrete-time system model, described by:

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

$$y(k) = Cx(k), \tag{2}$$

where $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ represents the sampling time, $x(k) \in \mathbb{R}^n$ represents the state vector, $u(k) \in \mathbb{R}^m$ is the control input and $y(k) \in \mathbb{R}^p$ is the measured output. The effect of measurement noise is not considered in this model, although possible (see Dórea (2009); Almeida and Dórea (2021)). It will be assumed that the state x and control u are restricted to the convex and compact (closed and bounded) sets containing the origin $\Omega_x \subset \mathbb{R}^n$ and $\mathfrak{U} \subset \mathbb{R}^m$, respectively, defined by:

$$\Omega_x = \{x : G_x x \leq \bar{1}\}, \quad \mathfrak{U} = \{u : Uu \leq \bar{1}\}, \tag{3}$$

where $G_x \in \mathbb{R}^{g_x \times n}$ and $U \in \mathbb{R}^{v \times m}$. $\bar{1}$ and $\bar{0}$ represent vectors (or matrices) of appropriate dimensions whose components are all equal to 1 and 0, respectively.

Definition 2.1 (Controlled Invariance (Blanchini, 1994)). *Given a contraction rate λ, $0 \leq \lambda < 1$, the set $\Omega \subset \Omega_x$ is said to be controlled-invariant λ-contractive w.r.t. system (1) if, $\forall x \in \Omega, \exists u \in \mathfrak{U} : Ax + Bu \in \lambda\Omega$, where $\lambda\Omega = \{\bar{x} : \bar{x} = \lambda x, x \in \Omega\}$.*

Definition 2.1 means that if $\Omega \subset \Omega_x$ is a controlled-invariant λ-contractive set, and once the initial condition $x(0) \in \Omega$, then there exists a state feedback control sequence $u(x(k)) \in \mathfrak{U}$ such that $x(k) \in \lambda\Omega$, $\forall k \geq 0$.

At this point, we look for a definition that characterizes a set as invariant under output feedback. To this end, consider next the set $\mathcal{Y}(\Omega) \subset \mathbb{R}^p$, which contains all admissible outputs y that can be associated to some $x \in \Omega$:

$$\mathcal{Y}(\Omega) = \{y : y = Cx \text{ for } x \in \Omega\}. \tag{4}$$

So, if $x \in \Omega$, then $y \in \mathcal{Y}(\Omega)$.

Definition 2.2 (OFCI Set (Dórea, 2009)). *Given a contraction rate λ, $0 \leq \lambda < 1$, the set $\Omega \subset \Omega_x$ is said to be Output-Feedback Controlled-Invariant (OFCI) w.r.t. system (1)-(2) if $\forall y \in \mathcal{Y}(\Omega), \exists u \in \mathfrak{U} : Ax + Bu \in \lambda\Omega, \forall x \in \Omega$ such that $Cx = y$.*

When Ω is OFCI, if $x(0) \in \Omega$, then one can compute a control sequence $u(y(k)) \in \mathfrak{U}$ such that $x(k) \in \lambda\Omega, \forall k \geq 0$. In (Dórea, 2009) necessary and sufficient conditions were proposed to verify whether a controlled-invariant set is OFCI with a contraction rate λ, from the solution of LP problems.

3. CONSTANT REFERENCE TRACKING: STATIC OUTPUT FEEDBACK CONTROL

Hereafter, we assume that $\Omega = \{x : Gx \leq \bar{1}\} \subset \Omega_x$, $G \in \mathbb{R}^{g \times n}$, is an OFCI polyhedron. The main objective is as follows: compute a sequence $u(k)$, $k = 0, 1, \ldots$, based on the measurements $y(k)$ such that the output y tracks a constant reference r, respecting at the same time all the imposed constraints, that is,

$$\forall x(0) \in \Omega : \begin{cases} \lim_{k \to \infty} y(k) = r, \\ x(k) \in \Omega, u(k) \in \mathfrak{U}, \forall k \geq 0 \end{cases}. \tag{5}$$

The set of admissible outputs (4) is also a convex and compact polyhedron defined by:

$$\mathcal{Y}(\Omega) = \{y : y = Cx \text{ for } x : Gx \leq \bar{1}\}. \tag{6}$$

Considering Definition 2.2, one can see that if Ω is OFCI with contraction rate λ, then

$$\begin{gathered} \forall y \in \mathcal{Y}(\Omega), \exists u \in \mathfrak{U} : G(Ax + Bu) \leq \lambda\bar{1}, \; Uu \leq \bar{1} \\ \forall x : Cx = y, Gx \leq \bar{1}. \end{gathered} \tag{7}$$

Since the state vector is not measured, we need to determine a single control $u \in \mathfrak{U}$ that keeps in Ω any x consistent with the measured output y. This can be achieved by calculating the worst case x, row by row of G, that may occur. Let the elements of the vector $\phi(y) \in \mathbb{R}^g$, which depend on the measurement y, be given by:

$$\begin{aligned} \phi_j(y) = \max_x \; & G_j A x, \quad j = 1, \ldots, g \\ \text{s.t.} \quad & Gx \leq \bar{1}, \quad Cx = y. \end{aligned} \tag{8}$$

Note that $\phi(y)$ is obtained for each measurement y. Hence, condition (7) can be rewritten as:

$$\forall y \in \mathcal{Y}(\Omega), \exists u \in \mathfrak{U} : \phi(y) + GBu \leq \lambda\bar{1}, \; Uu \leq \bar{1} \tag{9}$$

or

$$\forall y \in \mathcal{Y}(\Omega), \exists u \in \mathfrak{U} : \begin{bmatrix} \phi(y) \\ \bar{0} \end{bmatrix} + \begin{bmatrix} GB \\ U \end{bmatrix} u \leq \begin{bmatrix} \lambda\bar{1} \\ \bar{1} \end{bmatrix}. \tag{10}$$

Inequalities (10) are relevant because once satisfied, one can ensure the invariance of the polyhedron Ω. Next, we must find conditions that enable reference tracking.

The desired setpoint r is supposed to be such that the corresponding steady states do not violate the constraints $u \in \mathfrak{U}$ and $x \in \Omega$. A reference signal satisfying the above requirement is called admissible (Almeida et al., 2023). The tracking error is defined by $e(k) = r(k) - y(k)$, where $y(k)$ the measured output is assumed to be given by (2). The strategy to be considered is that of minimizing the one-step ahead absolute value of the error $|e(k+1)|$, which means to have the error in the next instant as small as possible. We assume $r(k+1) = r(k) = r$ (constant reference), then:

$$|e(k+1)| = |r - \underbrace{(CAx + CBu)}_{y(k+1)}| \leq \varepsilon \Rightarrow \begin{cases} CAx + CBu - r \leq \varepsilon \\ -CAx - CBu + r \leq \varepsilon \end{cases}. \tag{11}$$

$|e(k+1)|$ can be minimized by minimizing ε. The inequalities (11) can be rewritten as:

$$\begin{bmatrix} CA \\ -CA \end{bmatrix} x + \begin{bmatrix} CB & -\bar{1} \\ -CB & -\bar{1} \end{bmatrix} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \begin{bmatrix} r \\ -r \end{bmatrix}. \tag{12}$$

Consider the vector $\gamma(y) \in \mathbb{R}^{2p}$ given by:

$$\begin{aligned} \gamma_{\bar{j}}(y) = \max_x \; & \begin{bmatrix} CA \\ -CA \end{bmatrix}_{\bar{j}} x, \quad \bar{j} = 1, \ldots, 2p \\ \text{s.t.} \quad & Gx \leq \bar{1}, \quad Cx = y. \end{aligned} \tag{13}$$

The computation of $\gamma(y)$ considers the worst case x w.r.t. the minimization of ε. Condition (12) is equivalent to:

$$\begin{bmatrix} CB & -\bar{1} \\ -CB & -\bar{1} \end{bmatrix} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \bar{r} - \gamma(y), \tag{14}$$

where $\bar{r} = [r \;\; -r]^T$. Combining conditions (10) and (14), it is possible to compute the control action $u(y)$ in such a way that the system tracks the reference r satisfying simultaneously state and control constraints:

$$\begin{aligned} u(y(k)) = \; & \arg\min_{u,\varepsilon} \varepsilon \\ \text{s.t.} \; & \left[\begin{array}{cc} GB & \bar{0} \\ U & \bar{0} \\ \hline CB & -\bar{1} \\ -CB & -\bar{1} \end{array}\right] \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \left[\begin{array}{c} \lambda\bar{1} - \phi(y(k)) \\ \bar{1} \\ \hline \bar{r} - \gamma(y(k)) \end{array}\right] \end{aligned} \tag{15}$$

The control (15) is calculated online for each current measurement $y(k)$. The same is true for the vectors $\phi(y(k))$ and $\gamma(y(k))$.

4. THE COUPLED TANKS PROCESS

The system used here is the Quanser's Coupled Tanks (Figure 1), a widely employed solution in fuel storage and distribution, chemical industry, and water treatment sectors. This system is an excellent example to illustrate the dynamism of a flexible and widely used process.

Figure 1. The Quanser's Coupled Tanks.

Figure 2 shows a schematic diagram of the process. The process consists of two interconnected water tanks and one pump. Its input is the voltage $v(t)$ applied to the pump. $h_1(t)$ and $h_2(t)$ are the water levels of tanks 1 and 2, respectively. We consider the water level in tank 2, $h_2(t)$, as the system output.

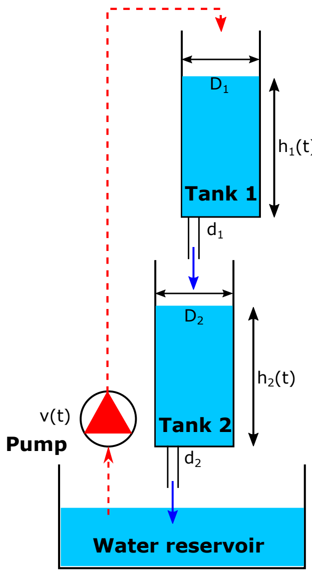

Figure 2. Schematic diagram of the coupled tanks system.

4.1 System Model

Here, we derive a mathematical model for the coupled tanks process from physics. Naturally, the model results in a state space model. The majority of systems are defined by nonlinear differential equations of the form:

$$\dot{h}(t) = f(h,v),\ s(t) = g(h,v),\ h \in \mathbb{R}^n,\ v \in \mathbb{R}^m,\ s \in \mathbb{R}^p. \quad (16)$$

In practice, one designs feedback controllers for (16) by reducing the problem to one of designing controllers for state-space linear systems that approximate it. Next, we consider a local linearization of (16) around an equilibrium point (Hespanha, 2018).

Definition 4.1 (Equilibrium). *A pair $(h^{eq}, v^{eq}) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium point of (16) if $f(h^{eq}, v^{eq}) = 0$. In this case,*

$$v(t) = v^{eq}, \quad h(t) = h^{eq}, \quad s(t) = s^{eq} = g(h^{eq}, v^{eq}), \quad \forall t \geq 0,$$

is a solution of (16).

Introducing the deviation variables $x = h - h^{eq}$, $u = v - v^{eq}$, and $y = s - s^{eq}$ the model (16) in the linearized form is then given by the following.

Definition 4.2. *The linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t) \quad (17)$$

defined by the following matrices:

$$A = \left[\left(\frac{\partial f_i}{\partial h_j}\right)_{ij}\right] \in \mathbb{R}^{n\times n}, B = \left[\left(\frac{\partial f_i}{\partial v_j}\right)_{ij}\right] \in \mathbb{R}^{n\times m},$$

$$C = \left[\left(\frac{\partial g_i}{\partial h_j}\right)_{ij}\right] \in \mathbb{R}^{p\times n}, D = \left[\left(\frac{\partial g_i}{\partial v_j}\right)_{ij}\right] \in \mathbb{R}^{p\times m},$$

all evaluated at (h^{eq}, v^{eq}), is called the local linearization of (16) around the equilibrium point (h^{eq}, v^{eq}).

The phenomenological model of the process can be obtained by application of mass balances and Bernoulli's law:

$$\begin{cases} \dot{h}_1(t) = -\frac{a_1}{A_1}\sqrt{2gh_1(t)} + \frac{k_m}{A_1}v(t) \\ \dot{h}_2(t) = \frac{a_1}{A_2}\sqrt{2gh_1(t)} - \frac{a_2}{A_2}\sqrt{2gh_2(t)} \\ s(t) = h_2(t) \end{cases} \quad (18)$$

where A_i is the cross-section (area) of tank i ($i = 1,2$), a_i is the cross-section of the outlet hole, and h_i is the water level. Note that in model (18), we assumed the corresponding pump flow as $q(t) = k_m v(t)$, where k_m is a constant. The acceleration of gravity is denoted g. The height of each tank is 30 cm. The parameter values of the process are given in Table 1. D_i and d_i (outlet hole), $i = 1,2$, stand for diameters as indicated in Figure 2.

Table 1. Parameter values of the process.

Parameter	Value	Unit
D_1, D_2	4.445	cm
d_1, d_2	0.476	cm
k_m	4.1	cm^3/Vs
g	981	cm/s^2

According to Definition 4.1, the equilibrium points of model (18) are obtained by setting $\dot{h}_1 = 0$ and $\dot{h}_2 = 0$, which results in:

$$h_1^{eq} = \left(\frac{k_m v^{eq}}{a_1}\right)^2 \frac{1}{2g}, \quad h_2^{eq} = \left(\frac{a_1}{a_2}\right)^2 h_1^{eq}, \quad s^{eq} = h_2^{eq}. \quad (19)$$

Then, in this case, it is possible to compute in a simple way h_1^{eq} and h_2^{eq} given the constant input v^{eq}. The linearized model of (18) around $(h_1^{eq}, h_2^{eq}, v^{eq})$, with $x_i = h_i - h_i^{eq}$, $i = 1,2$, $u = v - v^{eq}$, and $y = s - s^{eq}$, is then given by:

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{\dot{x}}=\underbrace{\begin{bmatrix}-\frac{a_1}{A_1}\sqrt{\frac{g}{2h_1^{eq}}} & 0\\ \frac{a_1}{A_2}\sqrt{\frac{g}{2h_1^{eq}}} & -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_2^{eq}}}\end{bmatrix}}_{A_c}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x}+\underbrace{\begin{bmatrix}\frac{k_m}{A_1}\\ 0\end{bmatrix}}_{B_c}u$$

$$y=\underbrace{[0\ \ 1]}_{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{20}$$

which can be rewritten in the more compact form as:

$$\begin{aligned}\dot{x}(t)&=A_cx(t)+B_cu(t)\\ y(t)&=Cx(t).\end{aligned} \tag{21}$$

The chosen operating point corresponds to $(h_1^{eq},h_2^{eq},v^{eq})=(15\,cm,15\,cm,7.4459\,V)$. One should notice that by doing this the point $(h_1^{eq},h_2^{eq})=(15\,cm,15\,cm)$ turns out to be the new system origin. The discrete-time equivalent to system (21), with empirically determined sample time $T_s=0.5$ seconds and zero-order hold as the discretization method, is:

$$A=e^{A_cT_s}=\begin{bmatrix}0.9677 & 0\\ 0.0317 & 0.9677\end{bmatrix},\ B=\int_0^{T_s}e^{A\tau}B_c\,d\tau=\begin{bmatrix}0.1300\\ 0.0021\end{bmatrix}.$$

5. RESULTS

Regarding the implementation of the constrained controller, a local server was implemented, which receives the sensor data from the plant, calculates the control signal and sends it to the plant. In particular, the corresponding calculations were implemented in part using Multi-Parametric Toolbox (MPT) for Matlab® software (Herceg et al., 2013). The server was implemented in Python (version 3.9.5), and plant communication takes place through the Simulink/Matlab® environment. Figure 3 illustrates the basic Simulink diagram.

The state constraints are established from the operating point, taking into account the maximum height of the tanks (30 cm).

$$\begin{matrix}\text{Tank 1}\\ h_{1_{min}}=0\end{matrix}:\begin{cases}x_{1_{min}}=h_{1_{min}}-h_1^{eq}=0-15=-15\,cm\\ x_{1_{max}}=h_{1_{max}}-h_1^{eq}=30-15=15\,cm\end{cases}\Longrightarrow$$

$$\begin{cases}x_{1_{min}}\le x_1\le x_{1_{max}}\\ -15\le x_1\le 15\end{cases}\Longrightarrow\begin{cases}x_1\le 15\\ -x_1\le 15\end{cases}\Longrightarrow\begin{cases}\frac{1}{15}x_1\le 1\\ -\frac{1}{15}x_1\le 1\end{cases}. \tag{22}$$

The same holds for tank 2: $\begin{cases}\frac{1}{15}x_2\le 1\\ -\frac{1}{15}x_2\le 1\end{cases}$. Then, the following polyhedron defining state constraints is obtained:

$$\underbrace{\begin{bmatrix}\frac{1}{15} & 0\\ -\frac{1}{15} & 0\\ 0 & \frac{1}{15}\\ 0 & -\frac{1}{15}\end{bmatrix}}_{G_x}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x}\le\underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}_{\bar{1}}\Longrightarrow\Omega_x=\{x:G_xx\le\bar{1}\}. \tag{23}$$

The maximum voltage that can be applied to the pump corresponds to $v(t)=12\,V$. Hence:

$$\begin{matrix}\text{Pump}\\ v_{min}=0\end{matrix}:\begin{cases}u_{min}=v_{min}-v^{eq}=-7.4459\,V\\ u_{max}=v_{max}-v^{eq}=4.5541\,V\end{cases}\Longrightarrow$$

$$\begin{cases}u_{min}\le u\le u_{max}\\ -7.4459\le u\le 4.5541\end{cases}\Longrightarrow\begin{cases}0.2196\,u\le 1\\ -0.1343\,u\le 1\end{cases}. \tag{24}$$

which can be written in the following polyhedral form:

$$\underbrace{\begin{bmatrix}0.2196\\ -0.1343\end{bmatrix}}_{U}u\le\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\bar{1}}\Longrightarrow\mathfrak{U}=\{u:Uu\le\bar{1}\}. \tag{25}$$

A λ-contractive controlled-invariant set Ω contained in the set of initial constraints Ω_x with a contraction rate of $\lambda=0.99$ was computed by using the algorithm proposed in (Dórea and Hennet, 1999). It was verified to be OFCI using the conditions proposed in (Dórea, 2009), with $\lambda=0.99$. The chosen contraction rate guarantees the largest possible invariant set. Therefore, the control sequence (15) can be implemented. The sets Ω_x and Ω are shown in Figure 4.

For the experimental results, we consider the following situation: initially, starting at the operating point, the system has to follow a constant reference $r=20\,cm$; then, after 160 seconds the reference changes to $r=23\,cm$. Figure 4 also depicts the trajectory of the state vector resulting from the control action (15), starting from the initial state $(h_1^{eq},h_2^{eq})=(15\,cm,15\,cm)$ which is labeled as $x(0)=[0\ 0]^T$. Due to calibration issues of the sensors, the initial state is slightly outside the origin.

In Figures 5 and 6 respectively, the corresponding time evolution of the output $y(k)=h_2(k)=x_2(k)+h_2^{eq}$ and the control signal $v(k)=u(k)+v^{eq}$ are shown. In addition, Figure 5 contemplates the evolution to the operating point of the system at which the controller is turned on. Although the control action $u(k)$ is based only on the measurement of tank 2, we have access to the measurement of tank 1. Figure 7 shows its response.

A state feedback version of the control (15) was also implemented. In this case, the vectors $\phi(y)$ in (8) and $\gamma(y)$ in (13) are replaced by $GAx(k)$ and $\begin{bmatrix}CA\\ -CA\end{bmatrix}x(k)$, respectively, since the complete state vector $x(k)$ is now available. Figures 8 to 11 portray the corresponding results.

From the presented results it is possible to see that the control action (15) delivered reference tracking at the cost of a considerable variability of the control signal, which can be attributed to the significant amplitude and frequency of the measurement noise. Another point to be touched is that we base the control actions on a discretized model coming from a continuous linearized one. Such a model is not guaranteed to capture the behavior of the system if it is far from the operating point. In our experiments we considered a setpoint of eight units away and even in this case the controller coped with its task. It can be observed that during the transient of the reference tracking the time response of tank 2 is smoother than that of tank 1, which is oscillatory with a high overshoot. The controller acts directly on tank 1, one way of increasing the input to tank 2 is rising the volume in tank 1. As stressed in (Dórea, 2009; Almeida and Dórea, 2021), by using the OFCI concept, it is possible to obtain solutions with larger sets of admissible initial states than well-established approaches. It can be observed in Figure 4 that the set of admissible initial states (Ω) is almost the set of state constraints (Ω_x).

Table 2 presents the average computational time required by both controllers to compute the control signal $u(k)$. One should notice that in the output feedback scenario $g+2p$ more LP problems need to be solved, which correspond to the computations of $\phi(y)$ and $\gamma(y)$.The results were obtained using an In-

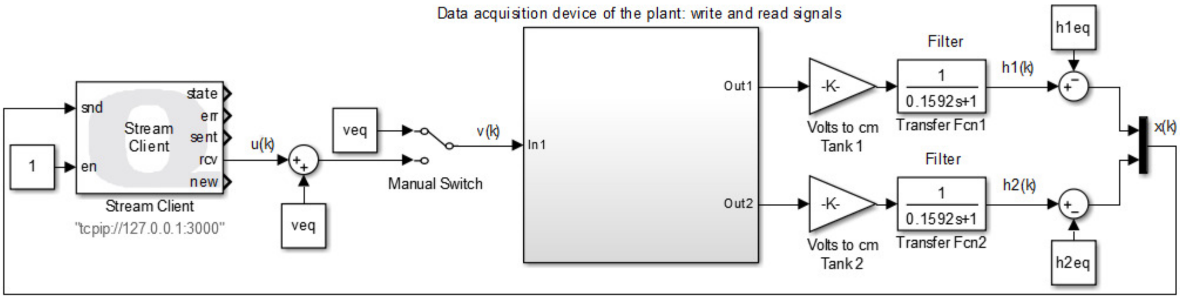

Figure 3. Simulink Block Diagram.

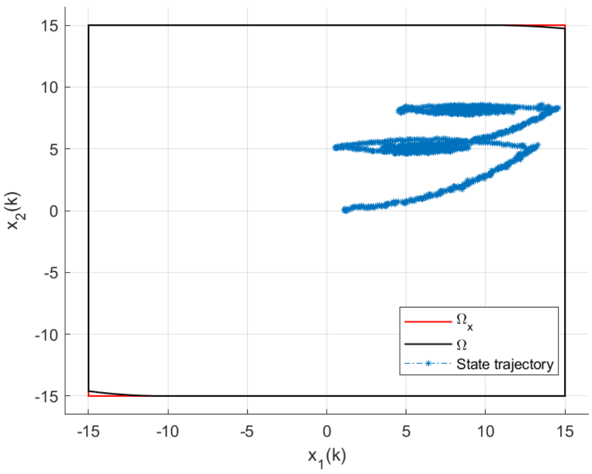

Figure 4. Set of state constraints Ω_x, OFCI set Ω, and state vector trajectory satisfying the constraints (*output feedback*).

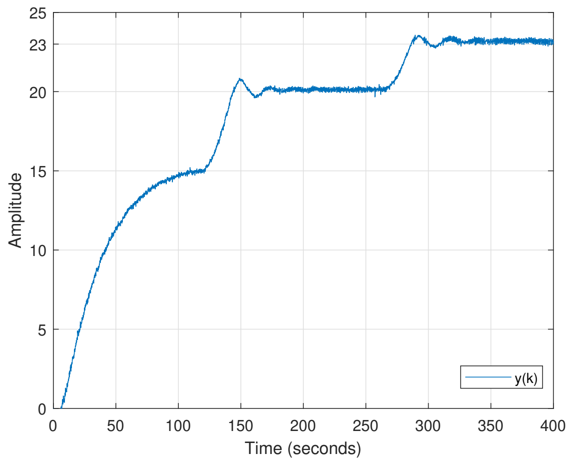

Figure 5. Time response $y(k)$: tank 2 (*output feedback*).

tel(R) Core(TM) $i5-4570$ CPU Processor $3.20\,GHz$, $24.0\,GB$ RAM, and $64-bit$ operating system.

Table 2. Average computational time per iteration.

Controller	Time (seconds)
State feedback	0.00099
Output Feedback	0.0143

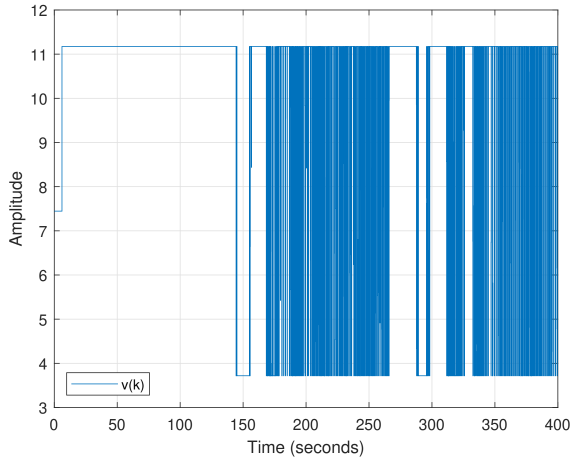

Figure 6. Control Signal $v(k)$ (*output feedback*).

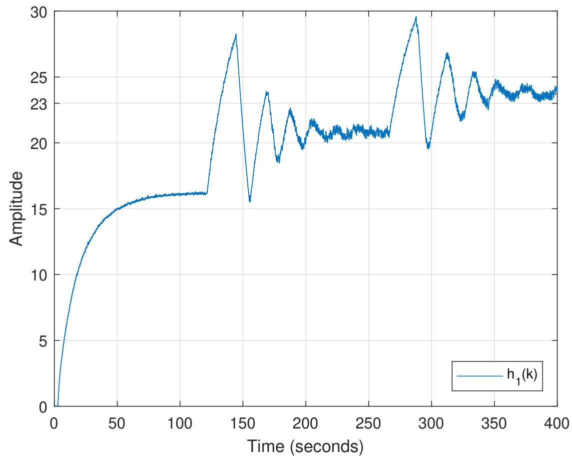

Figure 7. Time response of tank 1 (*output feedback*).

6. CONCLUSION

The design and implementation of output and state feedback controllers for constant reference tracking for constrained, linear, and discrete-time systems were presented. We considered a coupled tanks process to validate the controllers. Based on the concepts of OFCI sets and a discrete linearized model, a suitable control sequence was computed online to enforce the constraints and achieve reference tracking. The experimental results showed that the controllers met their requirements at the cost of a high variability in the control signal, mainly due to the influence of the measurement noise.

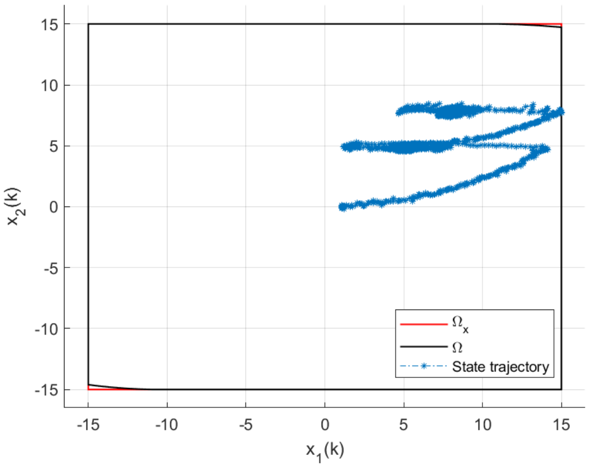

Figure 8. Set of state constraints Ω_x, controlled-invariant set Ω, and state-vector trajectory (*state feedback*).

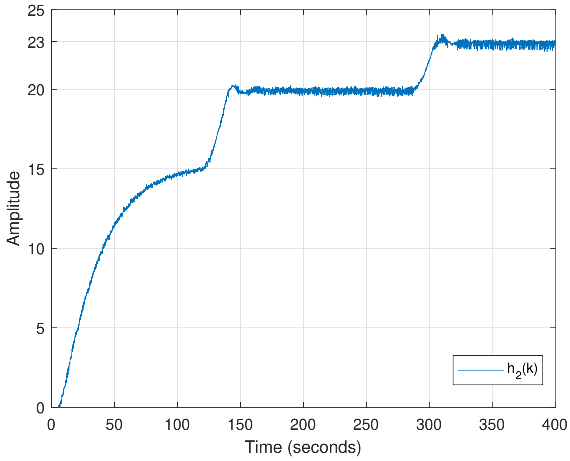

Figure 9. Time response $y(k)$: tank 2 (*state feedback*).

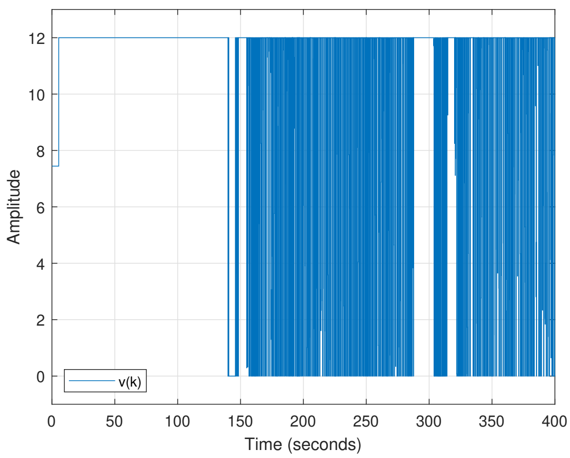

Figure 10. Control Signal $v(k)$ (*state feedback*).

Future works will address the achievement of smoother control signals and take into account the range of amplitude of noise. Also we intend to implement constrained dynamic controllers in physical systems of higher orders which use concepts of

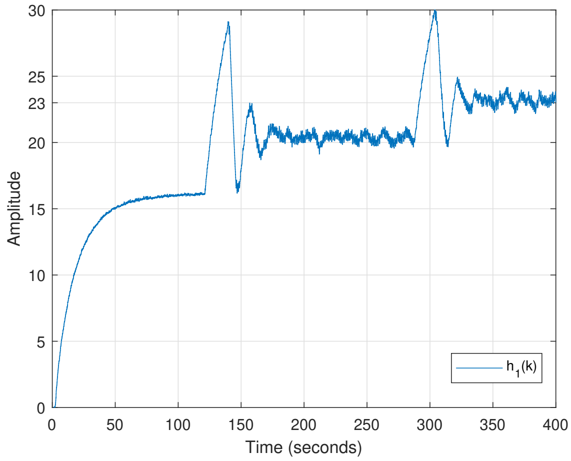

Figure 11. Time response of tank 1 (*state feedback*).

invariant sets w.r.t. estimation error. Furthermore, the effects of disturbances shall be considered and accounted for.

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