New Criterion for the Riemann Hypothesis

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June 7, 2023

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Abstract. There are several statements equivalent to the famous Riemann hypothesis. In 2011, Solé and and Planat stated that the Riemann hypothesis is true if and only if the inequality $\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. Using this result, we create a new criterion for the Riemann hypothesis. We prove the Riemann hypothesis is true using this new criterion.

Keywords: Riemann hypothesis · Prime numbers · Chebyshev function · Riemann zeta function.

1 Introduction

Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 1. *It is known that[1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number).

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. We know the following property for the Chebyshev function and the nth prime number:

Proposition 2. *For $n \geq 2$ [3, Theorem 1.1 pp. 2]:*

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \cdot (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \cdot \log^2 n}).$$

Proposition 3. *For $n \geq 8602$ [6, Theorem B (1.11) pp. 219]:*

$$q_n \leq n \cdot (\log n + \log\log n - 0.9385).$$

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this inequality is:

Proposition 4. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers $q_n > 3$ if and only if the Riemann hypothesis is true [9, Theorem 4.2 pp. 5].*

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [7, (17.) pp. 54]. We know the following formula:

Proposition 5. *We have that [2, Lemma 2.1 (1) pp. 359]:*

$$\sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k}\right) = \gamma - B = H.$$

The following property is based on natural logarithm function:

Proposition 6. *[5, pp. 1]. For $x > -1$:*

$$\log(1 + x) \leq x.$$

Putting all together yields a proof for the Riemann hypothesis using the Chebyshev function.

2 Central Lemma

Lemma 1. *For two real numbers $y > x > e$:*

$$\frac{y}{x} > \frac{\log y}{\log x}.$$

Proof. We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$\begin{aligned}
\frac{\log y}{\log x} &= \frac{\log(x + \varepsilon)}{\log x} \\
&= \frac{\log\left(x \cdot (1 + \frac{\varepsilon}{x})\right)}{\log x} \\
&= \frac{\log x + \log(1 + \frac{\varepsilon}{x})}{\log x} \\
&= 1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x}
\end{aligned}$$

and

$$\frac{y}{x} = \frac{x+\varepsilon}{x} = 1 + \frac{\varepsilon}{x}.$$

We need to show that

$$\left(1 + \frac{\log(1+\frac{\varepsilon}{x})}{\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

which is equivalent to

$$\left(1 + \frac{\varepsilon}{x \cdot \log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

using the Proposition 6. For $x > e$, we have

$$\frac{\varepsilon}{x} > \frac{\varepsilon}{x \cdot \log x}.$$

In conclusion, the inequality

$$\frac{y}{x} > \frac{\log y}{\log x}$$

holds on condition that $y > x > e$. □

3 What if the Riemann hypothesis were false?

Theorem 1. *If the Riemann hypothesis is false, then there are infinitely many prime numbers q_n such that $\mathsf{Dedekind}(q_n)$ does not hold.*

Proof. The Riemann hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [9, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f was introduced in the Nicolas paper [8, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) =$

$\Omega_+(x^{-b})$ [8, Theorem 3 (c) pp. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

4 A Key Theorem

Theorem 2.

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k})\right) = \log(\zeta(2)) - H.$$

Proof. We obtain that

$$\begin{aligned}
\log(\zeta(2)) - H &= \log(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k^2 - 1)}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k - 1) \cdot (q_k + 1)}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) + \log(\frac{q_k}{q_k + 1}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(\frac{q_k + 1}{q_k}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) \right) - \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) - \log(\frac{q_k}{q_k - 1}) + \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right)
\end{aligned}$$

by Propositions 1 and 5. □

5 A New Criterion

Theorem 3. $\mathsf{Dedekind}(q_n)$ *holds if and only if the inequality*

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is satisfied for the prime number q_n*, where the set* $S = \{x : \ x > q_n\}$ *contains all the prime numbers greater than* q_n *and* χ_S *is the characteristic function of the set* S *(This is the function defined by* $\chi_S(x) = 1$ *when* $x \in S$ *and* $\chi_S(x) = 0$ *otherwise).*

Proof. When $\mathsf{Dedekind}(q_n)$ holds, we apply the logarithm to the both sides of the inequality:

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > \gamma + \log\log\theta(q_n)$$

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

after of using the Theorem 2. Let's distribute the elements of the inequality to obtain that

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

when $\mathsf{Dedekind}(q_n)$ holds. The same happens in the reverse implication. □

6 The Main Insight

Theorem 4. *The Riemann hypothesis is true if the inequality*

$$\log(1 + \frac{1}{q_m}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n*, where* $m = \lfloor \frac{n}{4 \cdot \log^2 n} \rfloor$ *and* $\lfloor \ldots \rfloor$ *is the floor function.*

Proof. The inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_m\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is also satisfied, where the set $S' = \{x :\ x \geq q_m\}$ contains all the prime numbers greater than or equal to q_m (Note that $S \neq S'$ for large enough n). In the inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_m\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

only change the values of

$$\log(1 + \frac{1}{q_m}) + \log\log\theta(q_n)$$

and

$$\log\log\theta(q_{n+1})$$

between the consecutive primes q_n and q_{n+1}. It is enough to show that

$$\log(1 + \frac{1}{q_m}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers q_n. Indeed, the inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_m\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is the same as

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_{(m+1)}\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right)$$

$$> B + \log\log\theta(q_{n+1}) + \log(1 + \frac{1}{q_m}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1})$$

where q_n and q_{n+1} are consecutive primes. From the previous inequality, we note that if

$$\log(1 + \frac{1}{q_m}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1}) \geq 0$$

is satisfied, then

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_{(m+1)}\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_{n+1})$$

is also satisfied which means that $\mathsf{Dedekind}(q_{n+1})$ holds for large enough n according to the Theorem 3. Therefore, if the inequality

$$\log(1 + \frac{1}{q_m}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1}) \geq 0$$

is always satisfied starting for some large enough natural number n_0, (i.e. it is always satisfied for $n \geq n_0$), then we obtain that $\mathsf{Dedekind}(q_{n+1})$ always holds for $n \geq n_0$. However, this contradicts the fact that if the Riemann hypothesis is false, then there are infinitely many prime numbers q_{n+1} for which $\mathsf{Dedekind}(q_{n+1})$ does not hold when $n \geq n_0$. We obtain this contradiction as a consequence of the Theorem 1. By reductio ad absurdum, we have that the Riemann hypothesis is true when

$$\log(1 + \frac{1}{q_m}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. In fact, we only need to guarantee the existence of at least one large enough prime q_n where the inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_m\}}(q_k)) \cdot \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

holds precisely for $n \geq n_0$, such that we assume that $\mathsf{Dedekind}(q_{(m-1)})$ holds and the prime q_n implies that $\theta(q_n) < q_n - C \cdot \sqrt{q_n} \cdot \log\log\log q_n \leq \theta(q_{(m-1)})$ [4]. □

7 The Main Theorem

Theorem 5. *The Riemann hypothesis is true.*

Proof. The Riemann hypothesis is true when

$$\log(1+\frac{1}{q_m})+\log\log\theta(q_n)\geq\log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n because of the Theorem 4. That is the same as

$$\log(1+\frac{1}{q_m})\geq\log\log\theta(q_{n+1})-\log\log\theta(q_n)$$

$$\log(1+\frac{1}{q_m})\geq\log\left(\frac{\log\theta(q_{n+1})}{\log\theta(q_n)}\right)$$

after making the distribution. We would only need to prove that

$$1+\frac{1}{q_m}\geq\frac{\log\theta(q_{n+1})}{\log\theta(q_n)}$$

when we apply the exponentiation to the both sides. That is satisfied when

$$1+\frac{1}{q_n}\geq\frac{\theta(q_{n+1})}{\theta(q_n)}$$

since

$$\frac{\log\theta(q_{n+1})}{\log\theta(q_n)}<\frac{\theta(q_{n+1})}{\theta(q_n)}$$

by Lemma 1. By properties of the Chebyshev function, we have

$$1+\frac{1}{q_m}\geq 1+\frac{\log(q_{n+1})}{\theta(q_n)}$$

which is

$$q_m\leq\frac{\theta(q_n)}{\log(q_{n+1})}.$$

We know that

$$q_m\leq\frac{\theta(q_n)}{\log(q_{n+1})}$$

holds when

$$\frac{n}{4\cdot\log^2 n}\cdot(\log(\frac{n}{4\cdot\log^2 n})+\log\log(\frac{n}{4\cdot\log^2 n})-0.9385)\leq n\cdot(1-\frac{1}{\log n}+\frac{\log\log n}{4\cdot\log^2 n})$$

also holds by Propositions 2 and 3, where $\frac{n}{4\cdot\log^2 n}\geq m$. That would be equal to

$$(\log(\frac{n}{4\cdot\log^2 n})+\log\log(\frac{n}{4\cdot\log^2 n})-0.9385)\leq(4\cdot\log^2 n-4\cdot\log n+\log\log n)$$

which is trivially true for all sufficiently large prime numbers q_n. Consequently, we prove that the Riemann hypothesis is true. □

8 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

Acknowledgments

The author would like to thank Patrick Solé and Michel Planat for their support.

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