# Test for Uniform Bounded Input, Bounded Output Stability

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## 1 Introduction

Dynamical systems provide natural modes of various dynamic phenomena. From among dynamical systems, linear dynamical systems are tractable from analytical solution viewpoint. Thus, following the principle of occam's razor (principle of passimony), a sub-class of linear systems, called Linear Time Invariant (LTI) systems are used to provide tractable models of natural/artificial phenomena. In the analysis of LTI systems, the concept of "stability" is very crucial. Various types of stability such as $(i)$ Bounded Input-Bounded Output (BIBO) stability , $(ii)$ Lyapunov stability are utilized to study the dynamic behavior of LTI systems. Specifically stability tests such as Routh-Hurwitz stability are routinely used to determine BIBO stability. The concept of relative stability was also proposed.

In this research paper, we identify a limitation of BIBO stability and propose the concept of Uniform BIBO stability. A Stability test for Uniform BIBO stability (which utilizes the concept of relative stability) is discussed. It is realized that this novel concept of stability is of natural utility in the design of reliable voltage stabilizers and other interesting applications.

This research paper is organized as follows. In Section 2, known research literature is reviewed. In section 3, "uniform" BIBO stability of homogeneous (unforced) linear time invariant (LTI) system is discussed. In section 4, "uniform BIBO stability" of LTI system with state space description is discussed.

## 2 Related Research Literature: BIBO Stability $\rightarrow$ Uniform BIBO Stability

The concept of stability plays an important role in the design of linear control systems and other applications. One of the early concepts proposed for stability of LTI system is Bounded-Input, Bounded Output stability (BIBO). Formally we have that

**Definition 1.** *An LTI system is defined to be BIBO stable, if for every bounded input, the output is bounded.*

It is well known that a continuous time LTI system is BIBO stable if and only if the impulse response is absolutely integrable. Equivalently, the necessary and sufficient condition for continuous time LTI system to be BIBO stable is that all the poles of rational transfer function lie in the left half the $S-$plane (i.e. to the left of $j\omega$ axis). Also, for quantifying the degree of BIBO stability, the concept of relative stability is introduced by imposing the condition that poles of transfer function must lie to the left of aline parallel to $j\omega$ axis which is "strictly" to the left of it.

BIBO stability tests (which ca be suitably modified for relative stability) such as Routh-Hurwitz test are well studied. In this research paper, we introduce the concept of "UNIFORM BIBO stability" (which has practical applications) and discuss a relative stability based test to verify it for LTI continuous time systems.

**Definition 2.** *An LTI system is called UNIFORM BIBO stable if and only if for every bounded input, there exists a single contant, say $M$ which bounds the output in absolute value.*

*Note 1.* We consider Single input-Single output (SISO) continuous time LTI systems and derive uniform BIBO stability test. We realize that similar test can be derived for Discrete Time LTI systems and Multiple Input Multiple output (MIMO) systems. Details are avoided for brevity.

*Note 2.* All the poles of the SISO system are assumed to lie to the left of $j\omega$ axis i.e. we only consider BIBO stable LTI systems.

## 3 Homogeneous LTI System: Uniform BIBO Stability

Consider the Homogeneuous system

$$X'(t) = AX(t) \text{ for } t \geq 0, \tag{1}$$

where $A$ is an $N \times N$ matrix. The solution of (1) is given by $\mathbf{X}(t) = e^{At}\mathbf{X}(0)$. Assusming $A$ is diagonalizable then we have $A = \sum_{i=1}^{N} \lambda_i f_i g_i$ which can be rewritten as $A = \sum_{i=1}^{N} \lambda_i E_i$. Using the definition of matrix exponential one can easility obtain

$$e^A = \sum_{i=1}^{N} e^{\lambda_i} f_i g_i. \tag{2}$$

$$\text{and } \; e^{At} = \sum_{i=1}^{N} e^{\lambda_i t} f_i g_i. \tag{3}$$

Here $f_i$, $g_i$ are $N \times 1$ and $1 \times N$ matrices, respectively. Let $X(0)$ be the initial condition for the homogeneous system 1. Then the solution is $X(t) = e^{At}X(0)$. The bound on

$X(t)$ is given by

$$X(t) = \sum_{i=1}^{N} e^{\lambda_i t} E_i X(0) \tag{4}$$

$$\leq \sum_{i=1}^{N} e^{\lambda_1 t} E_i X(0) \tag{5}$$

$$\leq e^{\lambda_1 t} \sum_{i=1}^{N} E_i X(0) \tag{6}$$

If $A$ is diagonalizable we have $\sum_{i=1}^{N} E_i = I$, where $I$ is the identity matrix. Therefore

$$X(t) \leq e^{\lambda_1 t} X(0) \tag{7}$$

$$\leq X(0) \tag{8}$$

$$\leq max_i(X(0)_i) \tag{9}$$

Thus, the above bound on elements of initial state vector ensures uniform BIBO stability.

## 4 UNIFORM BIBO Stability of continuous time SISO, LTI systems

Consider the linear time varying case. The state space representation is given by

$$X'(t) = A(t)X(t) + B(t)u(t) \tag{10}$$

$$Y(t) = C(t)X(t) + D(t)u(t) \tag{11}$$

Now we consider LTI system. The state space representation is

$$X'(t) = AX(t) + Bu(t) \tag{12}$$

$$Y(t) = CX(t) + Du(t) \tag{13}$$

The solution of this system is

$$X(t) = e^{At}X(0) + \int_0^{\infty} e^{A(t-\tau)} Bu(\tau)d\tau \tag{14}$$

Suppose $X(t)$ is an $N \times 1$ vector and $Y(t)$ is an $M \times 1$ vector ( $\implies Moutputs$), $u(t)$ is an $L \times 1$ vector ( $\implies Linputs$) i.e. coefficient matrix $\{A(t), B(t), C(t), D(t)\}$ are of compatible dimensions.

It is well known that the transfer function of SISO, continuous time LTI system is given by

$$H(s) = L\{h(t)\} = C(sI - A)^{-1}B + D \tag{15}$$

where $h(t)$ is the impulse response matrix.

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(\tau) \tag{16}$$

$$Y(t) = CX(t) + Du(t) \tag{17}$$

SISO- case is easiest case

Investigate **uniformly, bounded input-bounded output stability**:

Goal: First bound components of $X(t)$ and then $y(t)$.

$$X(t) = e^{At}X(0) + \int_0^t \Big[\sum_{i=1}^{N} e^{\lambda_i(t-\tau)}E_i\Big]Bu(\tau)d\tau \tag{18}$$

$$= e^{At}X(0) + \sum_{i=1}^{N}\int_0^t \Big[e^{\lambda_i(t-\tau)}E_i\Big]Bu(\tau)d\tau \tag{19}$$

$$= e^{At}X(0) + \sum_{i=1}^{N}\Big[\int_0^t e^{\lambda_i(t-\tau)}u(\tau)d\tau\Big]E_iB \tag{20}$$

$$= e^{At}X(0) + \sum_{i=1}^{N}e^{\lambda_i t}\Big[\int_0^t e^{-\lambda_i\tau}u(\tau)d\tau\Big]E_iB \tag{21}$$

$$\leq \sup\{X(0)\} + \sum_{i=1}^{N}\Big[\int_0^\infty e^{\lambda_i(t-\tau)}u(\tau)d\tau\Big]E_iB \tag{22}$$

$$\leq \sup\{X(0)\} + \Big[\int_0^\infty e^{\lambda_1(t-\tau)}Md\tau\Big]B \tag{23}$$

$$\tag{24}$$

Let $\lambda_1^{'} = -\lambda_1$

$$X(t) \leq \sup\{X(0)\} + \frac{M}{\lambda_1^{'}}B \tag{25}$$

Let $\sup -norm(X_0), \sup -norm(B)$ be $\alpha, \beta$ respectively. Let $e = (1, 1, ..., 1)^T$. Then

$$X(t) \leq \alpha\mathbf{e} + \frac{M}{\lambda_1^{'}}\beta\mathbf{e} \tag{26}$$

Let $\gamma = max(\alpha, \beta)$.

$$X(t) \leq \gamma\mathbf{e} + \frac{M}{\lambda_1^{'}}\gamma\mathbf{e} \tag{27}$$

$$= \gamma\Big(1 + \frac{M}{\lambda_1^{'}}\Big)\mathbf{e} \tag{28}$$

Now let

$$\gamma\Big(1+\frac{M}{\lambda_1^{'}}\Big) < Q \tag{29}$$

$$\frac{M}{\lambda_1^{'}} < Q-\gamma \tag{30}$$

$$\frac{\lambda_1^{'}}{M} > \frac{1}{Q-\gamma} \tag{31}$$

$$\lambda_1^{'} > \frac{M}{Q-\gamma} = \theta \tag{32}$$

Output equation:

$$y(t) = CX(t) + du(t) \tag{33}$$

$$|y(t)| \leq |CX(t)| + |d|u(t) \tag{34}$$

$$= |CX(t)| + |d|M \tag{35}$$

$$\leq [L^1 - norm(C)]\eta + |d|M|y(t)| \leq \zeta \tag{36}$$

where $\zeta = [L^1 - norm(C)]\eta + |d|M \rightarrow$ Known ideas in LCS.

*Note 3.* Using the bound on the largest pole/eigen value (closest to $j\omega$ axis), well known relative stability test is employed to determine uniform BIBO stability test.

## 5 Formulation and solution in terms of Input-Output description: Uniform BIBO Stability

Let us consider simplest possible LTI system and its input-output description i.e. the rational transfer function has two real poles.

**Goal**: We want to determine bound on largest real pole such that the system is uniform BIBO stable.

Let $\alpha$ be the pole closer to $j\omega$ axis. $h(t) = c_1e^{-\alpha t} + c_2e^{-\beta t}$ : Real poles with $\alpha < \beta$.

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau \tag{37}$$

$$|y(t)| \leq M\int_0^t |h(\tau)|d\tau \tag{38}$$

We have

$$\int_0^t |h(\tau)|d\tau \leq c_1\int_0^\infty e^{-\alpha\tau}d\tau \tag{39}$$

$$\leq \frac{c_1}{\alpha} \tag{40}$$

Therefore uniform bound on output: $Q$.

$$\rightarrow M\tfrac{c_1}{\alpha} < Q$$
$$\rightarrow \tfrac{\alpha}{Mc_1} > \tfrac{1}{Q}$$
$$\alpha > \tfrac{Mc_1}{Q}$$

*Note 4.* The above result can easily be generalized to transfer function

$$H(s) = \frac{c_1}{s+\alpha} + \frac{c_2}{s+\beta} + ... \tag{41}$$

with poles and MIMO LTI systems.

## 6 Conclusions

In this research paper, the concept of uniform BIBO stability is introduced. This concept has significance in the design of real world control systems. A relative stability based test is proposed for determining the uniform BIBO stability of LTI systems.

[1], [2]

## References


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