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Abstract

The shallow water equations describe a thin layer of constant density fluid in hydrostatic balance, rotating or not**,** bounded from below by bottom topography and from above by a free surface. We construct the model of two layers shallow water flow with incline and uneven bottom. The governing equation of each layer consists of mass continuity, momentum equations, and the boundary conditions. The flat bottom topography typically chosen as b a constant variable such that the average of b is zero, and the uneven bottom topography typically chosen as function of space $b(x)$. The layers of the model are determined by the density of the fluid. The first layer is water exposed to sunlight which the temperature is higher than the second layer bellow it. The difference of density between the first and the second layer results in difference of gravity force which in the second layer we use reduced gravity force (gravity force decreases due to density ratio).

Keywords : Shallow water model, two layer, incline bottom, uneven bottom.

1 Introduction

Shallow water dynamics apply, by definition, to a fluid layer of constant density in which the horizontal scale of the flow is much greater than the layer depth. The shallow water equations describe a thin layer of constant density fluid in hydrostatic balance, rotating or not, bounded from below by a rigid surface and from above by a free surface, above which we suppose is another fluid of negligible inertia. The single-layer model is one of the simplest useful models in geophysical fluid dynamics, because it allows for a consideration of the effects of rotation in a simple framework without the complicating effects of stratification. By adding layers we can subsequently study the effects of stratification, and the model with two layers is not only a simple model of a stratified fluid, but also it is a surprisingly good model of many phenomena in the ocean and atmosphere [4].

Two-layer and multi-layer shallow water models are particularly useful in some limiting cases of multi-fluid and the density variable flows separated by nearly horizontal interfaces. These models govern the dynamics of incompressible fluids spreading under gravity effects. It can be for example water in the ocean can be considered as a two-layer fluid. The first layer is water that is exposed to sunlight the temperature is higher than the second layer. The second layer is the water below it [4].

The two-layer fluid flow model in the incline began was investigated by Pascal [3]. This model was applied to a flat bottom. The non-Newtonian fluid is used in the model is not too thick and applied to the mud flow in water. The single layer fluid flow model with the incline was developed by Yadav and Usha [5] for uneven bottom.

On the other hand, a shallow water flow model for flat bottom in the strait of Gilbraltar was derived by Chakir, et al [1] by considering wind stresses. In the special case, there is no mass exchange, assuming: the flow is quasi-horizontal, i.e. one can neglect the y component and the density is uniform in each layer. Chiapolino and Saurel [2] also derived two layer shallow water flow with some limitations with this approach: the vertical velocity component is neglected and the velocity is assumed uniform in cross sections of each layer.

In this research, we construct the models a of two layer shallow water flow with incline and uneven bottom. Boundary condition follows Chiapolino and Saurel [2].

This paper is organized as follows: mathematical formulation are consider in Section 2. Construction a two layer shallow water models with incline and uneven bottom are derived in Section 3. Non-dimensional the equations of system in Section 4. Conclusions are given in Section 5.

2 Mathematical Formulation

The problem of two layer shallow water flow with an inclined and uneven bottom is illustrated in Fig. 1 each the pressure of the upper layer is lower than the layer below it. The area in the first layer is $\eta_2(x,t) \leq z \leq \eta_1(x,t)$ and the second layer is $(x) \leq z \leq \eta_2(x,t)$, where $\eta_1(x,t)$ is the vertical displacement of free surface, $\eta_2(x,t)$ is the position of the interface, and $b(x)$ describes the uneven bottom topography. An incline bottom is affected by the angle θ, and h is the depth water.

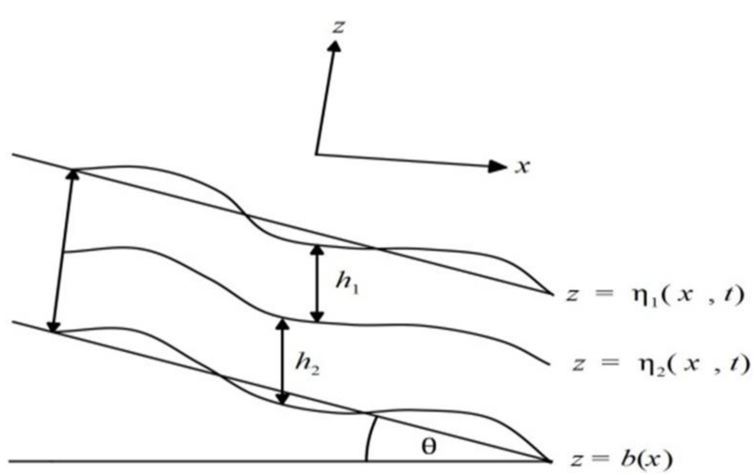

Fig. 1. A sketch of the two-layer model

The fluid motion is fully determined by mass continuity and the momentum equations. In this model, the fluid flows in x and z directions respectively, hence we have mass continuity of each layer as following

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \tag{1}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0. \tag{2}$$

In the momentum equation, the hydrostatics approximation is well statisfied since the vertical velocity is small enough. It is assumed the forces acting on the fluid volume element are only the surface force and body force. In actual circumstances, body force can be in the form of gravity force.

Forthemore for incline case, we can see in Fig. 2. which is for the gravity force f_g in x and z direction are

$$f_{g_x} = w \sin\theta,$$

$$f_{g_z} = -w \cos\theta.$$

where $w = mg$ is weight of an object, $m = \rho\Delta x\Delta y\Delta z$ denotes mass of object, and g is gravity.

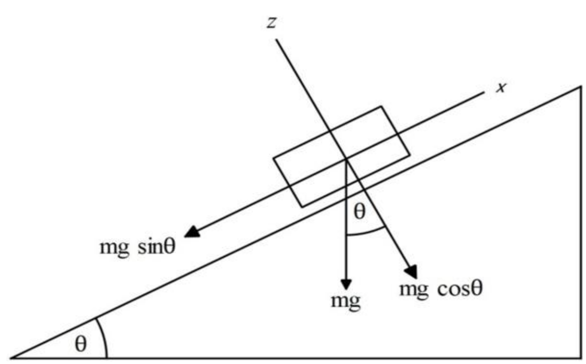

Fig. 2. Gravity force on the incline bottom

Therefore the body force in x and z directions respectively are given as follows

$$f_{g_x} = \rho g \sin\theta\, \Delta x\Delta y\Delta z,$$

$$f_{g_z} = -\rho g \cos\theta\, \Delta x\Delta y\Delta z.$$

where ρ is density. The momentum equation is

$$\rho_1\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + w_1\frac{\partial u_1}{\partial z}\right) = -\frac{\partial p_1}{\partial x} + \rho_1 g \sin\theta, \tag{3}$$

$$\rho_1\left(\frac{\partial w_1}{\partial t} + u_1\frac{\partial w_1}{\partial x} + w_1\frac{\partial w_1}{\partial z}\right) = -\frac{\partial p_1}{\partial x} - \rho_1 g \cos\theta, \tag{4}$$

$$\rho_2\left(\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + w_2\frac{\partial u_2}{\partial z}\right) = -\frac{\partial p_2}{\partial x} + \rho_2 g \sin\theta, \tag{5}$$

$$\rho_2\left(\frac{\partial w_2}{\partial t}+u_2\frac{\partial w_2}{\partial x}+w_2\frac{\partial w_2}{\partial z}\right)=-\frac{\partial p_2}{\partial x}-\rho_2 g\cos\theta, \quad (6)$$

where p denotes the pressure with $p_1 < p_2$, u_1 and u_2 are the components of the velocity along the x and z directions respectively and (f_{g_x}, f_{g_z}) are x and z the components of gravity force.

Since at the top of layer is the ambient fluid, the pressure on the fluid at the top is zero

$$p_1 = 0, \quad \frac{\partial u_1}{\partial z} = 0, \text{ at } \quad z = \eta_1(x,t), \quad (7)$$

where $z = \eta_1(x,t)$ is the position of the free surface of the first layer. At the interface between the two layers we have

$$p_1 = p_2, \quad \frac{\partial u_2}{\partial z} \quad \text{at} \quad z = \eta_2(x,t), \quad (8)$$

where $\mathrm{z} = \eta_2(\mathrm{x},\mathrm{t})$ is the position of the interface. We also have the condition of the velocity

$$u_1 = u_2, \quad w_1 = w_2 \quad \text{at} \quad z = \eta_2(x,t). \quad (9)$$

The kinematic conditions at the surface of the ambient fluid and at the interface between the two fluids can be expressed as

$$\frac{\partial \eta_1}{\partial t}+u_1\frac{\partial \eta_1}{\partial x} = w_1, \quad \text{at} \quad z = \eta_1(x,t), \quad (10)$$

$$\frac{\partial \eta_2}{\partial t}+u_2\frac{\partial \eta_2}{\partial x} = w_2, \quad \text{at} \quad \mathrm{z} = \eta_2(x,t), \quad (11)$$

$$u_2\frac{\partial b}{\partial x} = w_2, \quad \text{at} \quad z = b(x). \quad (12)$$

3 A two layer shallow water model of an incline and uneven bottom

In the shallow waters, the vertical velocity in the z direction is very small $\left(\frac{Dw}{Dt} \ll 0\right)$, such that equation (4) and (6) can be written as

$$0 = -\frac{\partial p_1}{\partial x} - \rho_1 g\cos\theta, \quad (13)$$

$$0 = -\frac{\partial p_2}{\partial x} - \rho_2 g\cos\theta. \quad (14)$$

In the each layer, pressure is given by the hydrostatic approximation. To get the pressure equations, we integrate equation (13) using the boundary conditions (7) . The equation of first layer is

$$\int \frac{\partial p_1}{\partial z} dz = -\int \rho_1 g\cos\theta\, dz,$$

$$p_1 = -\rho_1 g\cos\theta\, z + C,$$
$$p_1 = \rho_1 g\cos\theta\,(\eta_1 - z). \quad (15)$$

In the same way, we can find the pressure equation for the second layer by integrating equations (14) and using the boundary conditions (8)

$$p_2 = \rho_1 g \cos\theta\, \eta_1 - \rho_2 g \cos\theta\, z + g \cos\theta\, (\rho_2 - \rho_1)\eta_2. \tag{16}$$

The term involving z is irrelevant for the dynamics, because only the horizontal derivative enters the equation of motion. Inserting these expressions into the x-momentum equations (3) and (5) we yield

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + w_1 \frac{\partial u_1}{\partial z} = -g \cos\theta \frac{\partial \eta_1}{\partial x} + g \sin\theta, \tag{17}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + w_2 \frac{\partial u_2}{\partial z} = (g' - g) \cos\theta \frac{\partial \eta_1}{\partial x} - g' \cos\theta \frac{\partial \eta_2}{\partial x} + g \sin\theta, \tag{18}$$

where $g' = \frac{g(\rho_2 - \rho_1)}{\rho_2}$ is the reduced gravity (gravity force decreases due to density ratio).

Depth-integrating of the continuity equations (1) and (2), and the momentum equations (17) and (18), the Leibniz rule of integration is applied

$$\frac{\partial}{\partial x}\int_{\alpha(x)}^{\beta(x)} Q(x,y)\, dy = \int_{\alpha(x)}^{\beta(x)} Q(x,y)\, dy + Q\big(x,\beta(x)\big)\frac{\partial \beta(x)}{\partial x} - Q\big(x,\alpha(x)\big)\frac{\partial \alpha(x)}{\partial x}. \tag{19}$$

Therefore, the integral equations of (1), (2), (17), and (18) are written as

$$\frac{\partial}{\partial t}(\eta_1 - \eta_2) + \frac{\partial}{\partial x}\int_{\eta_2}^{\eta_1} u_1\, dz = 0, \tag{20}$$

$$\frac{\partial \eta_2}{\partial t} + \frac{\partial}{\partial x}\int_{b}^{\eta_2} u_2\, dz = 0, \tag{21}$$

$$\frac{\partial}{\partial t}\int_{\eta_2}^{\eta_1} u_1\, dz + \frac{\partial}{\partial x}\left(\int_{\eta_2}^{\eta_1} u_1^2\, dz + \frac{1}{2} g \cos\theta\, \eta_1^2\right) = g \sin\theta\, (\eta_1 - \eta_2) + g \cos\theta\, \eta_2 \frac{\partial \eta_1}{\partial x}, \tag{22}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{b}^{\eta_2} u_2\, dz + \frac{\partial}{\partial x}\left(\int_{b}^{\eta_2} u_2^2\, dz + \frac{1}{2} g \cos\theta\, \eta_1^2\right) \\ &\qquad = g \sin\theta\, (\eta_1 - b) + (g' - g) \cos\theta\, (\eta_2 - b)\frac{\partial \eta_1}{\partial x} + g' \cos\theta\, \eta_2 \frac{\partial \eta_1}{\partial x}. \end{aligned} \tag{23}$$

If we define

$\bar{u}_1 = \frac{1}{\eta_1 - \eta_2}\int_{\eta_2}^{\eta_1} u_1\, dz, \quad \bar{u}_2 = \frac{1}{\eta_2 - b}\int_{b}^{\eta_2} u_2\, dz,$

$q_1 = (\eta_1 - \eta_2)\bar{u}_1, \qquad q_2 = (\eta_2 - b)\bar{u}_2,$

where $\bar{u}_1$ and $\bar{u}_2$ are the depth-averaged velocities and q_1 and q_2 are water debit in the first and second layer respectively. The equations (20)-(23) can be expressed

$$\frac{\partial}{\partial t}(\eta_1 - \eta_2) + \frac{\partial q_1}{\partial x} = 0, \tag{24}$$

$$\frac{\partial \eta_2}{\partial t} + \frac{\partial q_2}{\partial x} = 0, \tag{25}$$

$$\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_1^2}{(\eta_1 - \eta_2)} + \frac{1}{2} g \cos\theta\, \eta_1^2\right) = g \sin\theta\, (\eta_1 - \eta_2) + g \cos\theta\, \eta_2 \frac{\partial \eta_1}{\partial x}, \tag{26}$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_2^2}{(\eta_2 - b)} + \frac{1}{2} g \cos\theta\, \eta_1^2\right)$$
$$= g \sin\theta\,(\eta_1 - b) + (g' - g)\cos\theta\,(\eta_2 - b)\frac{\partial \eta_1}{\partial x} + g' \cos\theta\, \eta_2 \frac{\partial \eta_1}{\partial x}. \quad (27)$$

Equations (24)-(27) variables called a two layer shallow water model with incline and uneven bottom in dimensional equation.

4 Non-dimensional the equations of system shallow water model

The units that we use to measure length, velocity and so on are irrelevant to the dynamics, and not necessarily the most appropriate units for a given problem. Rather, it is convenient to express the equations of motion, so far as is possible, in so-called non-dimensional variables, by which we express every variable (such as velocity) as the ratio of its value to some reference value. The observed surface waves have small H amplitudes and long L wavelengths, so it is necessary to look for shallow water equations on a scale variable. Scale is used to compare between real and actual circumstances with a model or picture. We now non-dimensionalized the equations of system (24)-(27). The x, z coordinates will be non-dimensionalized differently, recognizing the fact that there will be larger gradients in the vertical direction than the horizontal. To this end, we introduce the non-dimensional quantities

$$x = L\tilde{x}, \quad z = H\tilde{z}, \quad t = \frac{L}{U}\tilde{t}, \quad \eta_1 = H\tilde{\eta}_1,$$
$$\eta_2 = H\tilde{\eta}_2, \quad b = H\tilde{b} \quad q_1 = UH\tilde{q}_1, \quad q_2 = UH\tilde{q}_2,$$

where

$$U = (Hg')^{\frac{1}{2}}, \;\; L = \frac{U^2}{g}, \;\; \gamma = \frac{g'}{g}.$$

Employing the non-dimensional quantities into equations (24) - (27) and dropping the tildes as non-dimensional variables, we obtain

$$\frac{\partial}{\partial t}(\eta_1 - \eta_2) + \frac{\partial q_1}{\partial x} = 0, \quad (28)$$

$$\frac{\partial \eta_2}{\partial t} + \frac{\partial q_2}{\partial x} = 0, \quad (29)$$

$$\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_1^2}{(\eta_1 - \eta_2)} + \frac{1}{2\gamma}\cos\theta\, \eta_1^2\right) = \sin\theta\,(\eta_1 - \eta_2) + \frac{\cos\theta}{\gamma}\eta_2 \frac{\partial \eta_1}{\partial x}, \quad (30)$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_2^2}{(\eta_2 - b)} + \frac{1}{2}\cos\theta\, \eta_1^2\right)$$
$$= \sin\theta\,(\eta_1 - b) + \left(1 - \frac{1}{\gamma}\right)\cos\theta\,(\eta_2 - b)\frac{\partial \eta_1}{\partial x} + \cos\theta\, \eta_2 \frac{\partial \eta_1}{\partial x}. \quad (31)$$

Equation (28) - (31) is non-dimensional from of two layers shallow water model with incline and uneven bottom.

5 Conclusions

In this paper we have constructed the model of shallow water with two layer, we derived it based on its physical phenomenon that is with incline and uneven bottom. The governing equation of each layer consists of mass continuity and momentum equations. The layers of the model are determined by the density of the fluid. The difference of density between the first and the second layer, results in differences of gravity force which in the second layer we use reduced gravity force. The pressure at the first is layer smaller than the pressure at the second layer ($p_1 < p_2$). The two layer shallow water equation is obtained by changing the variable of dimensional to non-dimensional.

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