The Extended Fuzzy-Valued Convex Functions and Application

Tofigh Allahviranloo, Mohammad Reza Balooch Sharyari and Omolbanin Sedaghatfar

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T. Allahviranloo, M. R. Balooch Shahryari, O. Sedaghatfar

***Abstract*— The extended fuzzy-valued convex functions are considered in this paper. Also, some of the significant concepts such as the fuzzy indicator function, the fuzzy epigraph, the fuzzy infimal convolution, and the directional generalized derivative for extended fuzzy-values convex function are introduced and established.**

Keywords: Fuzzy numbers; The extended fuzzy-valued convex function; The fuzzy indicator function; The fuzzy epigraph; The fuzzy infimal convolution.

I. Introduction

Since Zadeh [11] began to study the basic concepts and principles of fuzzy theory, many studies have focused on the theoretical and practical aspects of fuzzy numbers. Zadeh proposed the concept of fuzzy numbers, so fuzzy numbers have been extensively researched by many researchers. For instance, Diamond and Kloeden [18], Dubois and Prade [4], Ghill et al. [2], Puri and Ralescu [16], Wang et al. [7], Wang and Wu [8], and Wu and Ma [3] discussed the problem of Hukuhara differentiability (H-differentiability for short), integrability, and measurability of fuzzy mapping. The fuzzy convex analysis is one of the most principal concepts in fuzzy optimization. Nada and Kar proposed the concept of convexity for fuzzy mapping in 1992 [19]. Accordingly, various studies on convexity for fuzzy mapping and application in fuzzy optimization have been organized [5], [6], [9], [10], [20]. Yan-Xu has explored the concepts of convexity and quasi-convexity of fuzzy-valued functions [10]. Syau has studied the concepts of quasi-convex and pseudo-convex multi-variable fuzzy functions [22]. Convexity and Lipschitz continuity of fuzzy-valued functions have been discussed by Furukawa [17]. Accordingly, some definitions for various kinds of convexity or generalized convexity of fuzzy mapping have been proposed and their properties have been studied [5], [21], [6]. Noor has been expressed the concept and properties of fuzzy preinvex functions in the $\mathbb{R}$ field [15]. A generalization of Hukuhara difference (H-difference), called the generalized Hukuhara difference (gH-difference), was proposed by Stefanini in 2010 because the H-difference exists between two fuzzy numbers only under very confined positions [12]. Compared to the H-difference, the gH-difference exists in more cases but does not always exist. To solve this problem, Bede and Stefanini introduced the generalized difference (g-difference), which always exists [1]. It should be noted that this difference in some cases does not maintain the convexity condition of fuzzy numbers, and therefore is not a fuzzy number. So this is determined by considering the convex hull of the resulting set by Gomes and Barros [14]. Based on these two differences, Bede and Stefanini have been obtained the generalized Hukuhara differentiability (gH-differentiability), and generalized differentiability (g-differentiability) [1]. In this paper, the concept of extended fuzzy-valued convex functions is introduced. Furthermore, we present some of the important concepts such as the fuzzy indicator function, the fuzzy epigraph, the fuzzy infimal convolution, and directional generalized deraivative for extended fuzzy-values are introduced and established.

T. Allahviranloo, Faculty of engineering and natural sciences, Istinye University, Istanbul, Turkey, (email: tofigh.allahviranloo@istinye.edu.tr)

M. R. Balooch Shahryar (Corresponding author), Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran,(email: baloochshahryari.1980@gmail.com)

O. Sedaghatfar, Department of Mathematics, Yadegar-e-Imam Khomeini (RAH) Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran, (email: sedaghatfar@yahoo.com)

II. Preliminaries

Definition 1. [13] With $\mathcal{K}_C$ the family of all closed and bounded intervals in $\mathbb{R}$ is demonstrated, i.e.,

$$\mathcal{K}_C = \left\{[\mathrm{a}^-, \mathrm{a}^+] \in \mathbb{R} \text{ and } \mathrm{a}^- \leq \mathrm{a}^+\right\}.$$

Definition 2. [19] For each $a \in \mathbb{R}$ can be considered as a fuzzy number $\tilde{a}$ defined by

$$\tilde{a}(t) = \begin{cases} 1, & t = a, \\ 0, & t \neq a. \end{cases}$$

$\mathbb{R}$ can be embedded in $\mathbb{R}_{\mathcal{F}}$.

Definition 3. Let us consider $-\tilde{\infty}$ and $+\tilde{\infty} \in \mathbb{R}_{\mathcal{F}}$ such that $-\tilde{\infty}(x) = 1$, if $x = -\infty$ and $-\tilde{\infty} = 0$, if $x \neq -\infty$, also, $+\tilde{\infty}(x) = 1$, if $x = +\infty$ and $+\tilde{\infty}(x) = 0$, if $x \neq +\infty$.

Definition 4. [8] Suppose that $X, Y \in \mathbb{R}_{\mathcal{F}}$, write $X \preceq Y \Leftrightarrow [X]_r = [X_r^-, X_r^+] \leq [Y]_r = [Y_r^-, Y_r^+]$ for every $r \in [0, 1]$. If $[X]_r \leq [Y]_r \Leftrightarrow X_r^- \leq Y_r^-$ and $X_r^+ \leq Y_r^+$. And $X \prec Y \Leftrightarrow X \preceq Y$ and $X \neq Y$.

Definition 5. [1] The gH-difference of two fuzzy numbers $X, Y \in \mathbb{R}_{\mathcal{F}}$ is defined by the form

$$X \ominus_{gH} Y = Z \Longleftrightarrow \begin{cases} \quad (1) & X = Y \oplus Z, \\ \text{or } (2) & Y = X \oplus (-1)Z. \end{cases}$$

Definition 6. [1] Suppose that $x_0 \in (\mathrm{a}, \mathrm{b})$ and γ with $x_0 + \gamma \in (\mathrm{a}, \mathrm{b})$, then the gH-derivative of a function $f : (\mathrm{a}, \mathrm{b}) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at x_0 is described as

$$f'_{gH}(x_0) = \lim_{\gamma \to 0} \frac{1}{\gamma}[f(x_0 + \gamma) \ominus_{gH} f(x_0)]. \tag{1}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ fulfilling Eq.(1) exists, it is said to be f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition 7. [1] The g-difference of two fuzzy numbers $X, Y \in \mathbb{R}_{\mathcal{F}}$ is distinctive by its r-cuts as

$$[X \ominus_g Y]_r = cl\Big(conv \cup_{\beta \geq r} ([X]_\beta \ominus_{gH} [Y]_\beta)\Big), \ \ \forall r \in [0,1], \quad (2)$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[X]_\beta$ and $[Y]_\beta$.

Proposition 1. *[1] The g-difference Eq.(2) for every $r \in [0,1]$ is given by*

$$[X \ominus_g Y]_r = \Big[\inf_{\beta \geq r} \min \Big\{ X_\beta^- - Y_\beta^-, X_\beta^+ - Y_\beta^+ \Big\}, \sup_{\beta \geq r} \ \max \Big\{ X_\beta^- - Y_\beta^-, X_\beta^+ - Y_\beta^+ \Big\} \Big].$$

Remark 1. [1] Assume that $X \ominus_{gH} Y \in \mathbb{R}_{\mathcal{F}}$ as well as $X \ominus_{gH} Y = X \ominus_g Y$.

Definition 8. [1] Suppose that $X \in \mathbb{R}_{\mathcal{F}}$ be a fuzzy number, for $r \in (0,1]$. The Hausdorff distance on $\mathbb{R}_{\mathcal{F}}$ is described by

$$D(X,Y) = \sup_{r \in [0,1]} \Big\{ \| [X]_r \ominus_{gH} [Y]_r \|_* \Big\},$$

where, for an interval $[x,y]$, the norm is

$$\| [x,y] \|_* = max\{|x|, |y|\}.$$

Proposition 2. *[1] For every $X, Y \in \mathbb{R}_{\mathcal{F}}$ we have $D(X,Y) = \sup_{r \in [0,1]} \| [X]_r \ominus_{gH} [Y]_r \|_* = \| X \ominus_g Y \|$, where $\|.\| = D(.,0)$.*

Definition 9. [1] Suppose that $x_0 \in (\mathrm{a}, \mathrm{b})$ and γ with $x_0 + \gamma \in (\mathrm{a}, \mathrm{b})$, then the level-wise gH-derivative (L_{gH}-derivative for short) of a function $f : (\mathrm{a}, \mathrm{b}) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at x_0 is described as the set of interval-valued gH-derivatives if they exist,

$$f'_{L_{gH}}(x_0)_r = \lim_{\gamma \to 0} \frac{1}{\gamma} ([f(x_0 + \gamma)]_r \ominus_{gH} [f(x_0)]_r).$$

If $f'_{L_{gH}}(x_0)_r \in \mathcal{K}_C \ \forall r \in [0,1]$, it is said to be f is L_{gH}-differentiable at x_0 and the family of intervals $\{f'_{L_{gH}}(x_0)_r | r \in [0,1]\}$ is the L_{gH}-derivative of f at x_0 and indicated by $f'_{L_{gH}}(x_0)$.

Definition 10. [1] Suppose that $x_0 \in (\mathrm{a}, \mathrm{b})$ and γ with $x_0 + \gamma \in (\mathrm{a}, \mathrm{b})$, then the g-derivative of a function $f : (\mathrm{a}, \mathrm{b}) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$f'_g(x_0) = \lim_{\gamma \longrightarrow 0} \frac{f(x_0 + \gamma) \ominus_g f(x_0)}{\gamma}. \quad (3)$$

If $f'_g(x_0) \in \mathbb{R}_{\mathcal{F}}$ fulfilling Eq.(3) exists, it is said to be f is g-differentiable at x_0.

Theorem 1. *[1] Suppose that $f : (\mathrm{a}, \mathrm{b}) \longrightarrow \mathbb{R}_{\mathcal{F}}$ be uniformly L_{gH}-differentiable at x_0. Then f is g-differentiable at x_0 and*

$$[f'_g(x_0)]_r = cl\Big(conv \cup_{\beta \geq r} f'_{L_{gH}}(x_0)_\beta\Big), \ \forall r \in [0,1]. \quad (4)$$

Theorem 2. *[1] Suppose that $f : [\mathrm{a}, \mathrm{b}] \longrightarrow \mathbb{R}_{\mathcal{F}}$ with $[f(x)]_r = [f_r^-(x), f_r^+]$. If the real-valued functions $f_r^-(x)$ and $f_r^+(x)$ both are differentiable w.r.t. x, uniformly w.r.t. $r \in [0,1]$, then $f(x)$ is g-differentiable and*

$$[f'_g(x)]_r = \Big[\inf_{\beta \geq r} \min \Big\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \Big\}, \sup_{\beta \geq r} \ \max \Big\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \Big\} \Big].$$

III. G-DIFFERENTIABILITY OF FUZZY-VALUED CONVEX FUNCTIONS

In this part, we take our fuzzy-valued function $f : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ to be defined on some interval of the real line $\mathbb{R}$. We mean to allow I to be open, half-open, or closed, finite or infinite. Also, we denote the interior of I by $int(\mathrm{I})$. Through this part, we discuss the g-differentiability and the basic facts about the g-differentiability properties of fuzzy-valued convex functions that can be easily visualized.

Definition 11. [8] Suppose that $f : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$, then f it is said to be a fuzzy-valued convex function if

$$f(\lambda x + (1-\lambda)y) \preceq \lambda \odot f(x) \oplus (1-\lambda) \odot f(y), \ \forall x, y \in \mathrm{I}, \quad \forall \ 0 \leq \lambda \leq 1.$$

Definition 12. Suppose that $f : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued convex function. Let $x \in \mathrm{I}$ and γ with $x + \gamma \in \mathrm{I}$ and $x - \gamma \in \mathrm{I}$ then,

$$f'_{+g}(x) = \lim_{\gamma \longrightarrow 0^+} \frac{f(x+\gamma) \ominus_g f(x)}{\gamma},$$
$$f'_{-g}(x) = \lim_{\gamma \longrightarrow 0^+} \frac{f(x-\gamma) \ominus_g f(x)}{\gamma},$$

exists on $int(\mathrm{I})$. We indicate that f is a fuzzy-valued convex function that right and left g-differentiable on $int(\mathrm{I})$.

Definition 13. Suppose that $f : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued convex function. Suppose that $x \in \mathrm{I}$ and γ with $x+\gamma$ and $x-\gamma \in \mathrm{I}$, the quotient $\mathcal{R}$ is right and left L_{gH}-difference for every $r \in [0,1]$, as below:

$$\mathcal{R}_{+L_{gH}}(x,\gamma)_r = \frac{[f(x+\gamma)]_r \ominus_{gH} [f(x)]_r}{\gamma},$$
$$\text{and } \mathcal{R}_{-L_{gH}}(x,\gamma)_r = \frac{[f(x)]_r \ominus_{gH} [f(x-\gamma)]_r}{\gamma}.$$

Note that, $\mathcal{R}_{+L_{gH}}(x,\gamma)_r$ and $\mathcal{R}_{-L_{gH}}(x,\gamma)_r \in \mathcal{K}_C \ \forall r \in [0,1]$. The collection of interval-valued $\{\mathcal{R}_{+L_{gH}}(x,\gamma)_r : r \in [0,1]\}$ and $\{\mathcal{R}_{-L_{gH}}(x,\gamma)_r : r \in [0,1]\}$ are right and left L_{gH}-quotient of f as a function of γ at x, and denoted by $\mathcal{R}_{+L_{gH}}(x,\gamma)$ and $\mathcal{R}_{-L_{gH}}(x,\gamma)$.

Theorem 3. *Suppose that $f : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued convex function, let $x \in int(\mathrm{I})$, γ with $x+\gamma$, $x-\gamma \in int(\mathrm{I})$. f is right and left uniformly L_{gH}-differentiable at x, then $f'_{+g}(x)$ and $f'_{-g}(x) \in \mathbb{R}_{\mathcal{F}}$ at every point of $int(\mathrm{I})$ exists, also $f'_{+g}(x) = \inf_{\gamma>0} \mathcal{R}_{+g}(x,\gamma)$, $f'_{-g}(x) = \sup_{\gamma>0} \mathcal{R}_{-g}(x,\gamma)$.*

IV. The extended fuzzy-valued convex functions

In before section,we were urged to consider fuzzy-valued convex functions with fuzzy-values in $\mathbb{R}_{\mathcal{F}}$. Henceforth we shall consider more general fuzzy-valued functions, with fuzzy-values in $\mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$.

Definition 14. A fuzzy-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is said to be convex, if for all $x, y, \lambda \in \mathbb{R}$ and $\gamma, \theta \in \mathbb{R}_{\mathcal{F}}$ such that $f(x) \prec \gamma$, $f(y) \prec \theta$, $0 < \lambda < 1$

$$f\Big(\lambda x + (1-\lambda)y\Big) \prec \lambda \odot \gamma \oplus (1-\lambda) \odot \theta.$$

Lemma 1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function then f is convex in Definition 11 if and only if f is convex fuzzy-valued function in Definition 14.*

Definition 15. The effective domain of a fuzzy-valued convex function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$, denoted by

$$dom(f) := \Big\{x \in \mathbb{R} : f(x) \prec +\tilde{\infty}\Big\}.$$

Lemma 2. *The effective domain of a fuzzy-valued convex function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is a convex set.*

Definition 16. A fuzzy-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is called fuzzy proper, if $f(x) \neq -\tilde{\infty}$, $\forall x \in \mathbb{R}$ and $f \not\equiv +\tilde{\infty}$.

Definition 17. A fuzzy-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is called fuzzy improper, if f is not fuzzy proper, i.e., $f(x) \equiv +\tilde{\infty}$ or $\exists\, x \in \mathbb{R} \ni f(x) = -\tilde{\infty}$.

The below theorem, is the class of improper fuzzy-valued convex functions which is easy to describe.

Theorem 4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ be an improper fuzzy-valued convex function. Then $f(x) = -\tilde{\infty}$ whenever $x \in int\big(dom(f)\big)$.*

The following Lemma deals with the case where an extended fuzzy-valued function becomes convex when restricted to a subset of its domain.

Lemma 3. **(The fuzzy extended fuzzy-value extension)** *Let $f : \mathrm{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued convex function, where I is a convex set. Define the fuzzy extension of f by*

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\},$$

$$\tilde{f}(x) := \begin{cases} f(x), & x \in \mathrm{I}, \\ +\tilde{\infty}, & x \notin \mathrm{I}. \end{cases} \tag{5}$$

Then $\tilde{f}$ is a fuzzy-valued convex function on $\mathbb{R}$, and $\tilde{f}$ is a fuzzy-valued convex extension of f to $\mathbb{R}$.

V. The fuzzy indicator function and the fuzzy epigraph

Now we introduce the fuzzy indicator function and the fuzzy epigraph of extended fuzzy-valued convex function as follows.

Definition 18. Let $C \subseteq \mathbb{R}^n$ be a set. Define the fuzzy indicator function of C by

$$\tilde{\mathrm{I}}_C : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\},$$

$$\tilde{\mathrm{I}}_C(x) := \begin{cases} \tilde{0}, & \text{if } x \in C, \\ +\tilde{\infty}, & \text{if } x \notin C. \end{cases}$$

Theorem 5. *Let $C \subseteq \mathbb{R}^n$. Then C is convex set if and only if $\tilde{\mathrm{I}}_C$ is fuzzy-valued convex function.*

Definition 19. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ be a fuzzy-valued function. The fuzzy epigraph $\big(Fepi(f) for\ short\big)$ of f is a subset $\mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ by

$$Fepi(f) := \Big\{(x, \tilde{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_{\mathcal{F}} : f(x) \preceq \tilde{\lambda}\Big\}.$$

Definition 20. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ be a fuzzy-valued function. The fuzzy strictly epigraph $\big(Fepi_s(f) for\ short\big)$ of f is a subset $\mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ by

$$Fepi_s(f) := \Big\{(x, \tilde{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_{\mathcal{F}} : f(x) \prec \tilde{\lambda}\Big\}.$$

Definition 21. A fuzzy-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ is said to be convex, if for all $x, y \in dom(f)$, $\lambda \in \mathbb{R}$ and $\gamma, \theta \in \mathbb{R}_{\mathcal{F}}$ such that $f(x) \prec \gamma$, $f(y) \prec \theta$, $0 < \lambda < 1$

$$f\Big(\lambda x + (1-\lambda)y\Big) \prec \lambda \odot \gamma \oplus (1-\lambda) \odot \theta.$$

Theorem 6. *Let f be a fuzzy-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$. The following conditions are equivalent:*

(1) *f is fuzzy-valued convex function.*
(2) *$Fepi(f)$ is convex set.*
(3) *$Fepi_s(f)$ is convex set.*

Definition 22. Let $A \subset \mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ be a convex set. We define a fuzzy-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ by

$$f(x) := \inf\Big\{\tilde{\lambda} \in \mathbb{R}_{\mathcal{F}} : (x, \tilde{\lambda}) \in A\Big\}, \quad \forall x \in \mathbb{R}^n.$$

Theorem 7. *Let $A \subset \mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ be a convex set, and let*

$$f(x) := \inf\Big\{\tilde{\lambda} \in \mathbb{R}_{\mathcal{F}} : (x, \tilde{\lambda}) \in A\Big\}, \quad \forall x \in \mathbb{R}^n.$$

Then f is a fuzzy-valued convex function.

VI. Fuzzy infimal convolution

Now, we introduce the fuzzy infimal convolution is a subset $\mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ as follows.

Definition 23. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\}$ fuzzy-valued functions. Define the fuzzy infimal convolution f and g by

$$f \Box g : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}} \cup \{+\tilde{\infty}\} \cup \{-\tilde{\infty}\},$$

$$\big(f \Box g\big)(x) := \inf\Big\{\tilde{\lambda} \in \mathbb{R}_{\mathcal{F}} : (x, \tilde{\lambda}) \in Fepi(f) \oplus Fepi(g)\Big\}.$$

Note that if f, g are fuzzy-valued convex functions, then $Fepi(f)$, $Fepi(g)$ in $\mathbb{R}^n \times \mathbb{R}_{\mathcal{F}}$ are convex sets,

$$\begin{aligned}(f\Box g)(x) &= \inf\left\{\tilde{\lambda}\in\mathbb{R}_{\mathcal{F}}:(x,\tilde{\lambda})\in \mathrm{Fepi}(f)\oplus \mathrm{Fepi}(g)\right\}\\ &= \inf\left\{\tilde{\lambda}\in\mathbb{R}_{\mathcal{F}}:\exists\left(x_1,\tilde{\lambda}_1\right)\in \mathrm{Fepi}(f),\exists\left(x_2,\tilde{\lambda}_2\right)\in \mathrm{Fepi}(g):(x,\tilde{\lambda})=\left(x_1,\tilde{\lambda}_1\right)\oplus\left(x_2,\tilde{\lambda}_2\right)\right\}\\ &= \inf\left\{\tilde{\lambda}\in\mathbb{R}_{\mathcal{F}}:\tilde{\lambda}=\tilde{\lambda}_1\oplus\tilde{\lambda}_2, x=x_1+x_2, f(x_1)\preceq\tilde{\lambda}_1, g(x_2)\preceq\tilde{\lambda}_2)\right\}\\ &= \inf\left\{\tilde{\lambda}\in\mathbb{R}_{\mathcal{F}}:f(x_1)\oplus g(x_2)\preceq\tilde{\lambda}, x=x_1+x_2, x_1,x_2\in\mathbb{R}^n\right\}\\ &= \inf\left\{f(x_1)\oplus g(x_2): x=x_1+x_2, x_1,x_2\in\mathbb{R}^n\right\}=\inf\left\{f(y)\oplus g(x-y):y\in\mathbb{R}^n\right\}.\end{aligned}$$

Hence

$$\big(f\Box g\big)(x)=\inf\left\{f(y)\oplus g(x-y):y\in\mathbb{R}^n\right\},$$

which is analogous to the formula for fuzzy integral convolution

$$\big(f*g\big)(x)=\int_{-\infty}^{+\infty} f(y)\odot g(x-y)\ dy.$$

Example 1. Let $\mathbb{R}^n$ be a normed space. Let $C\subseteq\mathbb{R}^n$ be a non-empty convex set. Define $f(x):=\langle -1,0,1\rangle\odot d(x,C)$, $\forall x\in\mathbb{R}^n$, where the distance of x from C is defined by

$$d(x,C):=\langle -1,0,1\rangle\odot\inf_{y\in C} f\|x-y\|.$$

We show that d_C is a fuzzy-valued convex function. Since C is convex set, then $\tilde{\mathrm{I}}_C$ is a fuzzy-valued convex function. Let $f(t)=\langle -1,0,-1\rangle\odot\|t\|$, $\forall t\in\mathbb{R}^n$, therefore f is fuzzy-valued convex function. We have

$$\begin{aligned}d_C(x) =&\langle -1,0,-1\rangle\odot\inf_{y\in C}\|x-y\|\\ =&\inf_{y\in\mathbb{R}^n}\left\{\tilde{\mathrm{I}}_C(x)\oplus\langle -1,0,1\rangle\odot\|x-y\|\right\}\\ =&\inf_{y\in\mathbb{R}^n}\left\{\tilde{\mathrm{I}}_C(x)\oplus f(x-y)\right\}\\ =&\left(\tilde{\mathrm{I}}_C\Box f\right)(x),\quad\forall x\in\mathbb{R}^n.\end{aligned}$$

So $d_C=\tilde{\mathrm{I}}_C\Box f$. Hence d_C is fuzzy-valued convex function.

VII. Directional g-derivative with extended fuzzy-values

Now, we introduce directional g-derivative of fuzzy-valued convex with extended fuzzy-values, also we consider some properties related to the directional g-derivative.

Definition 24. Let f be a fuzzy-valued function $f:\mathbb{R}^n\longrightarrow\mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ and $x_0,x\in\mathbb{R}^n$. The directional g-derivative of f at x_0 in the direction x is defined by

$$f'_g(x_0,x):=\lim_{t\to 0^+}\frac{f(x_0,tx)\ominus_g f(x_0)}{t}. \qquad (6)$$

If $f'_g(x_0,x)\in\mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ satisfying Eq.(6) exists. Note that if it exists ($+\tilde{\infty}$ and $-\tilde{\infty}$ being allowed as limits).

Theorem 8. *Let $f:\mathbb{R}^n\longrightarrow\mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ be a fuzzy proper fuzzy-valued convex function and $x_0\in dom(f)$. Then:*

(1) *$f'_g(x_0,x)$ exists, $\forall x\in\mathbb{R}^n$.*

(2) *$f'_g(x_0,.)$ is fuzzy positively homogeneous and fuzzy-valued convex function.*

Proposition 3. *Let $f:\mathbb{R}^n\longrightarrow\mathbb{R}_{\mathcal{F}}\cup\{+\tilde{\infty}\}\cup\{-\tilde{\infty}\}$ be a fuzzy-valued convex function and $x_0=dom(f)$ be such that $[f(x)]_r=[f_r^-(x),f_r^+(x)]$. Assume that $f_r^-(x)$ and $f_r^+(x)$ are real-valued multi variables convex functions directional differentiable at x_0 in the direction of x, uniformly w.r.t. $r\in[0,1]$, then $f(x)$ is directional g-differentiable at x_0 in the direction of x, and we have*

$$f'_g(x_0,x)_r=\Big[\inf_{\beta\geq r}\min\{f'^-_\beta(x_0,x),f'^+_\beta(x_0,x)\},\ \sup_{\beta\geq r}\max\{f'^-_\beta(x_0,x),f'^+_\beta(x_0,x)\}\Big].$$

VIII. Conclusion

Bede and Stefanini in 2013 introduced the concept of g-difference [1], which was introduced to define the concept of the extended fuzzy-valued convex functions. Moreover, some of these significant concepts have been established, such as the fuzzy indicator function, the fuzzy epigraph, the fuzzy infimal convolution, and the directional g-derivative for the extended fuzzy-valued convex functions. It has been suggested that some research is going to be performed based on considering g-subdifferential, the extended fuzzy-valued convex function, and its usage in fuzzy-valued convex optimization.

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