Higher-Order Interpolation of Cosserat Beam Deformations

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Abstract

A Cosserat beam is a one-dimensional continuum whose deformation field is described by a curve in $SE(3)$. Let $\mathbf{C}(s) \in SE(3)$ denote the displacement of the cross section. The (left-invariant) strain field in body-fixed representation [1] is defined by the deformation measure $\hat{\boldsymbol{\chi}} : [0,L] \to se(3)$ that satisfies the kinematic relation

$$\mathbf{C}' = \mathbf{C}\hat{\boldsymbol{\chi}}. \tag{1}$$

The (right-invariant) strain in spatial representation is defined by the deformation measure satisfying $\mathbf{C}' = \hat{\boldsymbol{\chi}}^{s}\mathbf{C}$. In [2, 3] this was called base-pole generalized curvature. Relation (1) allows reconstructing the beam deformation from the strain, and thus serves as kinematic reconstruction equation.

Geometrically exact beam formulations were introduced that are formulated on $SE(3)$. A crucial element in such formulations is the interpolation of the spatial deformations of beam elements [4, 5]. This is also crucial in the area of soft robotics [6, 7, 8] where the forward and inverse kinematics problem must be solved for robotic manipulators that are made from highly flexible beam elements. In this context, the state of the art is to assume constant curvature and piecewise constant cross sections in quasistatic conditions. With these assumptions the deformation of a segment with length L is interpolated as $\mathbf{C}(s) = \exp(\frac{s}{L}\hat{\mathbf{X}}_0)$, with $\hat{\mathbf{X}}_0 = \log\left(\mathbf{C}_0^{-1}\mathbf{C}_L\right)$, with $\mathbf{C}_0 = \mathbf{C}(0), \mathbf{C}_L = \mathbf{C}(L)$. This expression is known as Spherical Linear Interpolation (SLERP) [9], when the beam kinematics is modeled in $SO(3) \times \mathbb{R}^3$, and as Screw Linear Interpolation (ScLERP) [10], when kinematics is (correctly) modeled on $SE(3)$. The linear interpolation is not sufficient, in particular for large deformation of long slender beams.

In this paper a cubic and quartic interpolation scheme is presented. The interpolation respects initial and terminal values of the body-fixed strain measure. These 3rd/4th-order interpolation scheme allows exactly reconstructing the displacement of a beam (with constant cross section or cross linearly changing cross sections) subjected to a general wrench applied at the beam. The displacement is represented as $\mathbf{C}(\bar{s}) = \mathbf{C}_0 \exp\hat{\mathbf{X}}^{[k]}(\bar{s})$, where $\hat{\mathbf{X}}^{[k]}(\bar{s})$ is the kth-order approximation. Assuming $\mathbf{X}(0) = \mathbf{0}$, the 3rd-order approximation is

$$\mathbf{X}^{[3]}(\bar{s}) = \left(3\bar{s}^2 - 2\bar{s}^3\right)\mathbf{X}_L + \bar{s}(1-\bar{s})^2\bar{\boldsymbol{\chi}}_0 + \bar{s}^2(\bar{s}-1)\,\mathbf{dexp}^{-1}_{-\hat{\mathbf{X}}_L}\bar{\boldsymbol{\chi}}_L \tag{2}$$

where $\mathbf{X}_L = \log\left(\mathbf{C}_0^{-1}\mathbf{C}_L\right)$, and $\bar{\boldsymbol{\chi}}_0 = \bar{\boldsymbol{\chi}}(0), \bar{\boldsymbol{\chi}}_L = \bar{\boldsymbol{\chi}}(L)$, with $\bar{\boldsymbol{\chi}} = L\boldsymbol{\chi}$, and $\bar{s} = s/L$. The 4th-order approximation additionally allows prescribing $\bar{\boldsymbol{\chi}}'$ at the beam boundaries. An additional expression is available for the case where $\mathbf{X}(0) \neq \mathbf{0}$ or $\mathbf{X}(L) \neq \mathbf{0}$. The latter is important for loaded beams. The presented equations are derived by higher-order approximation of the Magnus expansion of the (local) kinematic reconstruction equation $\boldsymbol{\chi} = \mathbf{dexp}_{-\hat{\mathbf{X}}}\mathbf{X}'$ [11].

It is shown that this parameterization is singularity free, and applicable for singularity avoiding handling of slender semi-deformable objects (SDLO), i.e. deformable element including rigid parts as connectors.

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