Eigenvalues and eigenvectors for order 3 symmetric matrices: An analytic approach

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ABSTRACT

It is always important to find the eigenvalues and eigenvectors accurately since they provide vital information to many engineering problems. Analytic solutions are advantageous over numerical solutions because numerical solutions are accurate up to some order while analytic solutions are exact. Again, the analytic solution of a polynomial of order greater than 3 is too complicated. In this article, we show simple, optimized, fast and robust analytic ways to find the eigenvalues and eigenvectors of symmetric matrices of order 3.

Keywords:
eigenvalues, eigenvectors, linear Algebra

Introduction

Physical systems can be expressed using sets of linear and nonlinear equations. In the linear or linearized models we explain the system behavior with the help of eigenvalues and corresponding eigenvectors. A majority of systems are be reduced to three dimensional space and can be modeled using linear or linearized equations. In these linear equations the dependent and independent vectors are coupled using a matrix called coupling matrix. This coupling matrix has different names i.e. in the stress-strain relationship we call it compliance matrix. Again we also represent the stress or strain on a body (see fig. 1) using a 3×3 matrix and the eigenvalues of this matrix express the principal stresses or strains and the eigenvectors express the directions of corresponding eigenvalues[1]. In the dynamical systems (linear or linearized) eigenvalues of the coupling matrix express and characterize the fixed points of the system[2]. Eigenvalues and eigenvectors also provide important information to many other problems like matrix diagonalization[3], vibration analysis[4], chemical reactions[5], face recognition[6], electrical networks[7], Markov chain model[8], atomic orbitals[9] and more.

Though analytical solutions are exact by nature but the general analytical methods for finding the eigenvalues and eigenvectors are limited to the matrices of order 4 since there is no explicit analyti-

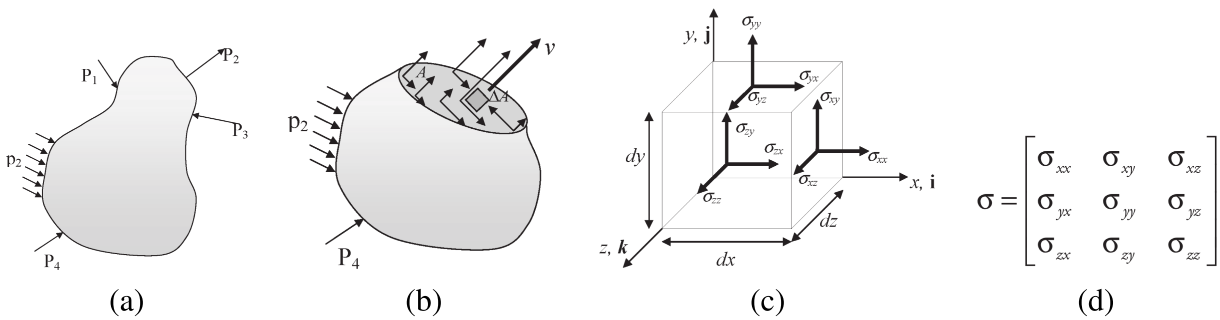

Figure 1: (a)- General 3D body with different loads acting on its surface. (b)- Free body diagram of (a). (c)- The stresses acting on the different faces of a cube of vanishing dimensions (dx, dy, and dz tending to zero) representing a point in the body, also called a *material point*. (d) The stress tensor.[1]

cal solution for a polynomial of order ≥ 5. Numerical methods works great to find the eigenvalues and eigenvectors with a certain accuracy. There are a number of efficient numerical algorithms available for finding the eigenvalues and eigenvectors. Some very popular algorithms are Power iteration method[10], Jacobi method[11] and QR[12]. There are also very efficient publicly available software packages for finding eigenvalues and eigenvectors i.e. LAPACK[13], GSL the GNU scientific library[14], ARPACK[15], Armadillo[16], NumPy[17], SciPy[18] or Intel MKL[19]. These softwares are optimized for larger matrices and using these softwares to find eigenvalues and eigenvectors of a 3×3 symmetric matrices, the computational overheads becomes prominent. In this article we propose an analytical routine to find the eigenvalues and eigenvectors efficiently in a robust way. We explain the theories associated to each steps of our algorithm show a couple of examples and present a flowchart of an implemented code.

Theory

We represent any 3x3 real valued symmetric matrix as,

$$[\mathrm{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

For any matrix of rank n (maximal number of independent column or row of a matrix), there exists n scalars λ_i for which eq. (1) is true where $i = 1 \cdots n$. In our case, $n = 3$ and matrix [A] is symmetric i.e. $\mathrm{A}_{ij} = \mathrm{A}_{ji}$.

$$[\mathrm{A}]\vec{v}_i = \lambda_i \vec{v}_i \tag{1}$$

We call the scalar value λ_i as the i-th eigenvalue and the vector $\vec{v}_i$ as the eigenvector associated to λ_i of matrix [A]. Re-writing eq. (1) and taking the determinant, the eigenvalues of matrix [A] can be found using eq. (2)

$$|\mathrm{A} - \lambda_i \mathrm{I}| = 0 \tag{2}$$

where, $|\cdot|$ represents the determinant of any matrix. Equation (2) is also known as the characteristic polynomial[20] of matrix [A]. We know from elementary linear algebra, the eigenvalues of a real symmetric matrix [A] are also real[3]. So we find all three real eigenvalues of 3x3 symmetric matrix just solving the cubic characteristic polynomial. The characteristic equation for this matrix can be obtained as,

$$\lambda^3 - \alpha\lambda^2 - \beta\lambda - \gamma = 0 \tag{3}$$

where,

$$\alpha = a_{11} + a_{22} + a_{33} \tag{4a}$$

$$\beta = a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{11}a_{22} - a_{22}a_{33} - a_{33}a_{11} \tag{4b}$$

$$\begin{aligned}\gamma =& a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{12}^2a_{33} \\ & - a_{13}^2a_{22}\end{aligned} \tag{4c}$$

α and γ are the trace and the determinant of matrix A respectively. Alternatively, β can be written as $\left\{ - \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} \right\}$. Again α, β and γ are also known as the *invariants*. Because these quantities α, β and γ are unique to all the matrices having the same eigenvalues. We solve the characteristic equation analytically[1] as follows,

$$p = -\left(\frac{3\beta + \alpha^2}{3}\right) \tag{5a}$$

$$q = -\left(\gamma + \frac{2\alpha^3}{27} + \frac{\alpha\beta}{3}\right) \tag{5b}$$

$$\cos\phi = -\frac{q}{2\sqrt{(|p|/3)^3}} \tag{5c}$$

$$\lambda_1 = \frac{\alpha}{3} + 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{\phi}{3}\right) \tag{6a}$$

$$\lambda_2 = \frac{\alpha}{3} - 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{\phi - \pi}{3}\right) \tag{6b}$$

$$\lambda_3 = \frac{\alpha}{3} - 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{\phi + \pi}{3}\right) \tag{6c}$$

In some cases, from the construction of matrix $[A]$ we can instantly determine some eigenvalues. For example, knowing λ_1 and using the facts in eq. (7), we can derive the alternate eq. (8) to determine the other two eigenvalues λ_2 and λ_3,

$$\lambda_1 + \lambda_2 + \lambda_3 = \alpha \tag{7a}$$

$$\lambda_1\lambda_2\lambda_3 = \gamma \tag{7b}$$

$$\lambda_{2,3} = \frac{\alpha - \lambda_1}{2} \pm \sqrt{\left(\frac{\alpha - \lambda_1}{2}\right)^2 - \frac{\gamma}{\lambda_1}} \tag{8}$$

We also can find λ_3 with knowledge of λ_1 and λ_2 using eqs. (7) and (9)

$$\lambda_3 = \alpha - \lambda_1 - \lambda_2 = \frac{\gamma}{\lambda_1 \lambda_2} \tag{9}$$

We can find the eigenvectors of matrix $[\mathrm{A}]$ for their corresponding eigenvalues re-writing eq. (1) as eq. (10) as follows.

$$[\mathrm{A} - \lambda_i \mathrm{I}]\vec{v}_i = \vec{0} \tag{10}$$

Assume λ_i and $\vec{v}_i$ are the eigenvalues and the eigenvectors of $[\mathrm{A}]$. Let $\vec{v}_i = \{l_i, m_i, n_i\}$.

We write eq. (10) explicitly as follows,

$$\begin{bmatrix} a_{11} - \lambda_i & a_{12} & a_{13} \\ a_{12} & a_{22} - \lambda_i & a_{23} \\ a_{13} & a_{23} & a_{33} - \lambda_i \end{bmatrix} \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} = 0 \tag{11}$$

Let, $[\mathrm{B}]_i = [\mathrm{A}] - \lambda_i[\mathrm{I}]$ and write the simplified linear equations of eq. (11) as

$$b^i_{11} + b_{12} m_i + b_{13} n_i = 0 \tag{12a}$$

$$b_{12} l_i + b^i_{22} m_i + b_{23} n_i = 0 \tag{12b}$$

$$b_{13} l_i + b_{23} m_i + b^i_{33} n_i = 0 \tag{12c}$$

Since, the sub-equations in eq. (12) are not linearly independent from each other, we can not find all the roots from the equations in eq. (12). So we need another condition and we consider that the condition to be the fact that $\vec{v}_i$ is a unit direction vector i.e.,

$$l_i^2 + m_i^2 + n_i^2 = 1 \tag{13}$$

In this study we find two eigenvectors solving equations (12) and (13) and the third eigenvector is from the cross product of the found two.
As we mentioned earlier, in some cases we can find some eigenvalues of matrix $[\mathrm{A}]$ making logical arguments. So to make our algorithm computationally efficient we specialized our analytical algorithm for the different cases as follows. Moreover any analytic solution is inherently efficient because it is a direct solution using substitution and thus requiring no iterations.

Case 1: Only diagonal elements are present in $[\mathrm{A}]$
The eigenvalues of the diagonal matrix are the diagonal elements[21] and the eigenvectors are simple i.e. $\{\{1,0,0\}, \{0,1,0\}, \{0,0,1\}\}$.

Case 2: Generalized plane stress or plane strain
There can be three sub-cases,

- Case 2.1: **If** $[\text{A}] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}$
  then, $\lambda_1 = a_{11}$ and the other two eigenvalues λ_2 and λ_3 can be found either from eq. (6) or eq. (8) or alternatively using Mohr's circle as below. The eigenvalues are,

  $$\lambda_1 = a_{11}$$

  $$\lambda_{2,3} = \frac{a_{22} + a_{33}}{2} \pm \sqrt{\left(\frac{a_{22} - a_{33}}{2}\right)^2 + a_{23}^2}$$

  and the eigenvectors are,

  $$\{\{1, 0, 0\}, \{0, -\frac{b_{23}}{\sqrt{b_{22}^{i2} + b_{23}^2}}, \frac{b_{22}^i}{\sqrt{b_{22}^{i2} + b_{23}^2}}\},$$
  $$\{0, -\frac{b_{22}^i}{\sqrt{b_{22}^{i2} + b_{23}^2}}, -\frac{b_{23}}{\sqrt{b_{22}^{i2} + b_{23}^2}}\}\}$$

- Case 2.2: **If** $[\text{A}] = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{bmatrix}$
  then, $\lambda_2 = a_{22}$ and the other two eigenvalues λ_1 and λ_3 can be found either from eq. (6) or (8) or alternatively using Mohr's circle as below. The eigenvalues are,

  $$\lambda_2 = a_{22}$$

  $$\lambda_{1,3} = \frac{a_{11} + a_{33}}{2} \pm \sqrt{\left(\frac{a_{11} - a_{33}}{2}\right)^2 + a_{13}^2}$$

  and the eigenvectors are,

  $$\{\{0, 1, 0\}, \{-\frac{b_{13}}{\sqrt{b_{11}^{i2} + a_{13}^2}}, 0, \frac{b_{11}^i}{\sqrt{b_{11}^{i2} + a_{13}^2}}\},$$
  $$\{\frac{b_{11}^i}{\sqrt{b_{11}^{i2} + a_{13}^2}}, 0, \frac{b_{13}}{\sqrt{b_{11}^{i2} + a_{13}^2}}\}\}$$

- Case 2.3: **If** $[\text{A}] = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
  then, $\lambda_3 = a_{33}$ and the other two eigenvalues λ_1 and λ_2 can be found either from eq. (6) or

eq. (8) or alternatively using Mohr's circle as below. The eigenvalues are,

$$\lambda_3 = a_{33}$$
$$\lambda_{1,2} = \frac{a_{11}+a_{22}}{2} \pm \sqrt{\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}^2}$$

and the eigenvectors are,

$$\{\{0,0,1\},\{-\frac{b_{12}}{\sqrt{b_{11}^{i2}+b_{12}^2}},\frac{b_{11}^i}{\sqrt{b_{11}^{i2}+b_{12}^2}},0\},$$
$$\{-\frac{b_{11}^i}{\sqrt{b_{11}^{i2}+b_{12}^2}},-\frac{b_{12}}{\sqrt{b_{11}^{i2}+b_{12}^2}},0\}\}$$

Case 3: General cases
We find the eigenvalues using eqs. (5) and (6) and to make our analytical algorithm computationally efficient and robust we solve the eigenvectors explicitly for all the possible cases of matrix $[\mathrm{A} - \lambda_i I]$.

- Case 3.1: $(b_{11}^i b_{23} - b_{13}b_{12})b_{13} \neq 0$ or $(b_{12}^2 - b_{11}^i b_{22}^i)b_{13} \neq 0$
- Case 3.2: $(b_{11}^i b_{33}^i - b_{13}^2)b_{12} \neq 0$ or $(b_{12}b_{13} - b_{11}^i b_{23})b_{12} \neq 0$
- Case 3.3: $(b_{12}b_{33}^i - b_{23}b_{13})b_{11}^i \neq 0$ or $(b_{22}^i b_{13} - b_{12}b_{23})b_{11}^i \neq 0$
- Case 3.4: $(b_{12}b_{23} - b_{13}b_{22}^i)b_{23} \neq 0$ or $(b_{11}^i b_{22}^i - b_{12}^2)b_{23} \neq 0$
- Case 3.5: $(b_{12}b_{33}^i - b_{13}b_{23})b_{22}^i \neq 0$ or $(b_{11}^i b_{23} - b_{12}b_{13})b_{22}^i \neq 0$
- Case 3.6: $(b_{22}^i b_{33}^i - b_{23}^2)b_{12} \neq 0$ or $(b_{12}b_{23} - b_{22}^i b_{13})b_{12} \neq 0$
- Case 3.7: $(b_{13}b_{22}^i - b_{12}b_{23})b_{33}^i \neq 0$ or $(b_{11}^i b_{23} - b_{13}b_{12})b_{33}^i \neq 0$
- Case 3.8: $(b_{13}b_{23} - b_{12}b_{33}^i)b_{23} \neq 0$ or $(b_{11}^i b_{33}^i - b_{13}^2)b_{23} \neq 0$
- Case 3.9: $(b_{23}^2 - b_{22}^i b_{33}^i)b_{13} \neq 0$ or $(b_{12}b_{33}^i - b_{23}b_{13})b_{13} \neq 0$

For brevity sake, we present the following three cases,

If: $(b_{11}^i b_{23} - b_{13}b_{12})b_{13} \neq 0$ **or** $(b_{12}^2 - b_{11}^i b_{22}^i)b_{13} \neq 0$
From $b_{12} \times$ (12a) $- b_{11}^i \times$ (12b) we can write

$$m_i = \frac{b_{11}^i b_{23} - b_{13}b_{12}}{b_{12}^2 - b_{11}^i b_{22}^i} n_i = \mathcal{Q}_i n_i \tag{14}$$
$$n_i = \frac{b_{12}^2 - b_{11}^i b_{22}^i}{b_{11}^i b_{23} - b_{13}b_{12}} m_i = \mathcal{R}_i m_i \tag{15}$$

From (14) and (12c) we can write,

$$l_i = -\frac{b_{23}\mathcal{Q}_i + b^i_{33}}{b_{13}} n_i = \mathcal{P}^n_i n_i \tag{16}$$

From (15) and (12c) we can write,

$$l_i = -\frac{b_{23} + b^i_{33}\mathcal{R}_i}{b_{13}} m_i = \mathcal{P}^m_i m_i \tag{17}$$

And using eq. (13) we can write as follows

$$\begin{aligned} n_i &= \frac{1}{\sqrt{\mathcal{P}^{n2}_i + \mathcal{Q}^2_i + 1}} \\ m_i &= \frac{1}{\sqrt{\mathcal{P}^{m2}_i + 1 + \mathcal{R}^2_i}} \end{aligned} \tag{18}$$

The eigenvector is either
$\left\{\frac{\mathcal{P}^n_i}{\sqrt{\mathcal{P}^{n2}_i+\mathcal{Q}^2_i+1}}, \frac{\mathcal{Q}_i}{\sqrt{\mathcal{P}^{n2}_i+\mathcal{Q}^2_i+1}}, \frac{1}{\sqrt{\mathcal{P}^{n2}_i+\mathcal{Q}^2_i+1}}\right\}$ or
$\left\{\frac{\mathcal{P}^m_i}{\sqrt{\mathcal{P}^{m2}_i+1+\mathcal{R}^2_i}}, \frac{1}{\sqrt{\mathcal{P}^{m2}_i+1+\mathcal{R}^2_i}}, \frac{\mathcal{R}_i}{\sqrt{\mathcal{P}^{m2}_i+1+\mathcal{R}^2_i}}\right\}$.
If: $(b^i_{11}b^i_{33} - b^2_{13})b_{12} \neq 0$ **or** $(b_{12}b_{13} - b^i_{11}b_{23})b_{12} \neq 0$
From $b_{13} \times$ (12a) $- b^i_{11} \times$ (12c) we can write

$$m_i = \frac{b^i_{11}b^i_{33} - b^2_{13}}{b_{12}b_{13} - b^i_{11}b_{23}} n_i = \mathcal{Q}_i n_i \tag{19}$$

$$n_i = \frac{b_{12}b_{13} - b^i_{11}b_{23}}{b^i_{11}b^i_{33} - b^2_{13}} m_i = \mathcal{R}_i m_i \tag{20}$$

From (19) and (12b) we can write,

$$l_i = -\frac{b^i_{22}\mathcal{Q}_i + b^i_{33}}{b_{12}} n_i = \mathcal{P}^n_i n_i \tag{21}$$

From (20) and (12b) we can write,

$$l_i = -\frac{b^i_{22} + b_{23}\mathcal{R}_i}{b_{12}} m_i = \mathcal{P}^m_i m_i \tag{22}$$

And using eq. (13) we can write as follows

$$\begin{aligned} n_i &= \frac{1}{\sqrt{\mathcal{P}^{n2}_i + \mathcal{Q}^2_i + 1}} \\ m_i &= \frac{1}{\sqrt{\mathcal{P}^{m2}_i + 1 + \mathcal{R}^2_i}} \end{aligned} \tag{23}$$

The eigenvector is either
$\left\{\frac{\mathcal{P}_i^n}{\sqrt{\mathcal{P}_i^{n2}+\mathcal{Q}_i^2+1}}, \frac{\mathcal{Q}_i}{\sqrt{\mathcal{P}_i^{n2}+\mathcal{Q}_i^2+1}}, \frac{1}{\sqrt{\mathcal{P}_i^{n2}+\mathcal{Q}_i^2+1}}\right\}$ or
$\left\{\frac{\mathcal{P}_i^m}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}, \frac{1}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}, \frac{\mathcal{R}_i}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}\right\}$.

If: $(b_{12}b_{33}^i - b_{23}b_{13})b_{11}^i \neq 0$ **or** $(b_{22}^i b_{13} - b_{12}b_{23})b_{11}^i \neq 0$
From $b_{13} \times$ (12b) $- b_{12} \times$ (12c) we can write

$$m_i = \frac{b_{12}b_{33}^i - b_{23}b_{13}}{b_{22}^i b_{13} - b_{12}b_{23}} n_i = \mathcal{Q}_i n_i \tag{24}$$

$$n_i = \frac{b_{22}^i b_{13} - b_{12}b_{23}}{b_{12}b_{33}^i - b_{23}b_{13}} m_i = \mathcal{R}_i m_i \tag{25}$$

From (24) and (12a) we can write,

$$l_i = -\frac{b_{12}\mathcal{Q}_i + b_{13}}{b_{11}^i} n_i = \mathcal{P}_i^n n_i \tag{26}$$

From (25) and (12a) we can write,

$$l_i = -\frac{b_{12} + b_{13}\mathcal{R}_i}{b_{11}^i} m_i = \mathcal{P}_i^m m_i \tag{27}$$

And using eq. (13) we can write as follows

$$\begin{aligned} n_i &= \frac{1}{\sqrt{\mathcal{P}_i^{n2} + \mathcal{Q}_i^2 + 1}} \\ m_i &= \frac{1}{\sqrt{\mathcal{P}_i^{m2} + 1 + \mathcal{R}_i^2}} \end{aligned} \tag{28}$$

The eigenvector is either
$\left\{\frac{\mathcal{P}_i^n}{\sqrt{\mathcal{P}_i^{n2}+\mathcal{Q}_i^2+1}}, \frac{\mathcal{Q}_i}{\sqrt{\mathcal{P}_i^{n2}+\mathcal{Q}_i^2+1}}, \frac{1}{\sqrt{\mathcal{P}_i^{n2}+\mathcal{Q}_i^2+1}}\right\}$ or
$\left\{\frac{\mathcal{P}_i^m}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}, \frac{1}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}, \frac{\mathcal{R}_i}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}\right\}$.

For the cases described in **Case 3: General cases** the multiplicity of eigenvalue can be 2 maximum meaning there will be at least 2 distinct eigenvalues, which can be explained using Mohr's circle if shearing is present ($a_{ij} \neq 0,\ i \neq j$). So using our above described methodology we are able to find all the eigenvalues and their corresponding eigenvectors.

Examples

We show below a couple of examples for different configurations of [A].
Example 1: Lets find the eigenvalues and eigenvectors of the following matrix

$$\begin{bmatrix} 0.6375 & 0 & -1.7835 \\ 0 & 7.1568 & 0 \\ -1.7835 & 0 & 1.4508 \end{bmatrix}$$

We find the eigenvalues using the equations (6a), (6b) and (6c) as $\{-0.7851,\ 7.1568,\ 2.8734\}$ and from Case 2 condition of the *Generalized Plain stress or plain strain* we determined the eigenvectors as,
$\{\{0.7818,\ 0,\ 0.6236\},\ \{0,\ 1,\ 0\},\ \{0.6236,\ 0,\ -0.7818\}\}$.

Example 2: Lets find the eigenvalues and eigenvectors of the following matrix

$$\begin{bmatrix} 0 & -1.0298 & 1.0792 \\ -1.0298 & 1.2554 & 0 \\ 1.0792 & 0 & 0.1547 \end{bmatrix}$$

Similarly, using the equations (6a), (6b) and (6c) we find the eigenvalues as $\{-1.2514,\ 0.6442,\ 2.0172\}$ and following *General cases* we find the eigenvectors as,
$\{\{0.7542,\ 0.3098,\ -0.5789\},\ \{0.3390,\ 0.5713,\ 0.7475\},\ \{-0.5623,\ 0.7600,\ -0.3258\}\}$.
We have verified our results with Matlab 2018b[22].

The Algorithm

We find all the eigenvalues using eqs. (4), (5), and (6) or alternatively using eqs. (8) or (9) or Mohr's circle. At first we determine the case of matrix [A] and then follow the algorithm as per the steps described for the associated case to find first two eigenvectors and the third eigenvector from the cross product of first two eigenvectors. We show the flowchart of our algorithm in fig. 2.

Conclusion

In this article we have presented the theory and algorithm to find the eigenvalues and eigenvectors of a 3×3 symmetric matrix. Our algorithm can be an efficient approach to find the eigenvalues and eigenvectors for the symmetric matrices of order 3 over the publicly available software packages since they are difficult to use. Our method is simple, optimized, very robust and can easily be implemented in any programming language even in a handheld calculator.

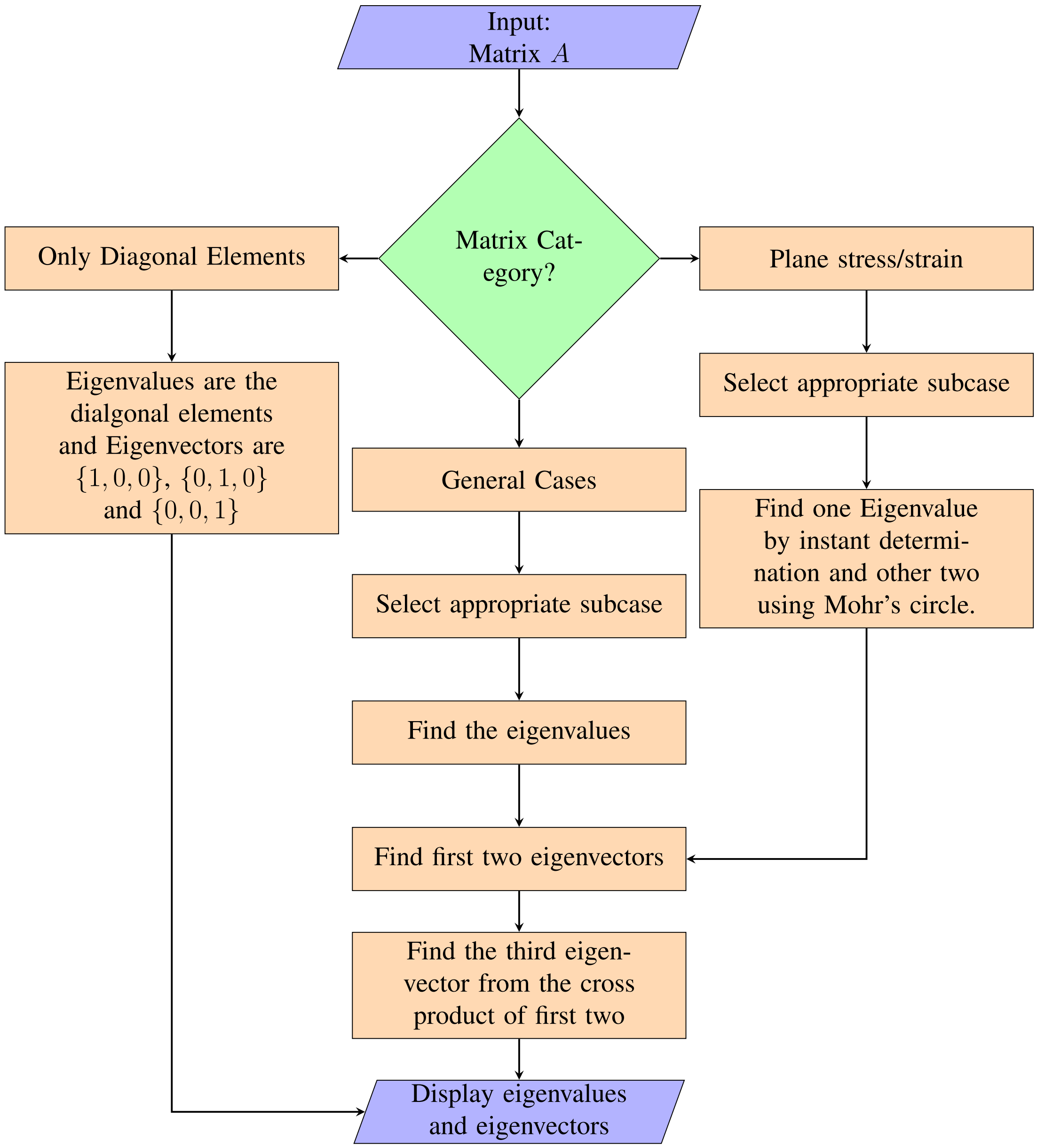

Figure 2: Flowchart of the algorithm

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References

[1] T. A. Khraishi and Y.-L. Shen, *Introductory Continuum Mechanics with Applications to Elasticity*. Cognella, revised ed., 2013.

[2] J. M. T. Thompson and H. B. Stewart, *Nonlinear dynamics and chaos*. John Wiley & Sons, 2 ed., 2002.

[3] C. R. Howard Anton, *Elementary Linear Algebra: Applications Version*. John Wiley, 11 ed., 2013.

[4] L. Meirovitch, *Elements of vibration analysis*. McGraw-Hill Companies, 1975.

[5] R. Marcus, "On the analytical mechanics of chemical reactions. quantum mechanics of linear collisions," *The Journal of Chemical Physics*, vol. 45, no. 12, pp. 4493–4499, 1966.

[6] M. A. Turk and A. P. Pentland, "Face recognition using eigenfaces," in *Computer Vision and Pattern Recognition, 1991. Proceedings CVPR'91., IEEE Computer Society Conference on*, pp. 586–591, IEEE, 1991.

[7] F. Dörfler and F. Bullo, "Kron reduction of graphs with applications to electrical networks.," *IEEE Trans. on Circuits and Systems*, vol. 60, no. 1, pp. 150–163, 2013.

[8] S. Karlin, *A first course in stochastic processes*. Academic press, 2014.

[9] W. J. Hehre, R. F. Stewart, and J. A. Pople, "self-consistent molecular-orbital methods. i. use of gaussian expansions of slater-type atomic orbitals," *The Journal of Chemical Physics*, vol. 51, no. 6, pp. 2657–2664, 1969.

[10] S. C. Chapra and R. P. Canale, *Numerical methods for engineers*, vol. 2. Mcgraw-hill New York, 1998.

[11] R. Mathias, "Accurate eigensystem computations by jacobi methods," *SIAM journal on matrix analysis and applications*, vol. 16, no. 3, pp. 977–1003, 1995.

[12] E. F. Anley, "The qr method for determining all eigenvalues of real square matrices," *Pure and Applied Mathematics Journal*, vol. 5, no. 4, pp. 113–119, 2016.

[13] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen, *LAPACK Users' Guide*. Philadelphia, PA: Society for Industrial and Applied Mathematics, third ed., 1999.

[14] M. Galassi *et al.*, *GNU Scientific Library Reference Manual*. Network Theory Ltd, 3rd ed., 2016.

[15] R. B. Lehoucq, D. C. Sorensen, and C. Yang, *ARPACK users' guide: solution of large-scale eigenvalue problems with implicitly restarted Arnoldi methods*, vol. 6. Siam, 1998.

[16] C. Sanderson and R. Curtin, "Armadillo: a template-based c++ library for linear algebra," *Journal of Open Source Software*, 2016.

[17] T. E. Oliphant, *Guide to NumPy*, vol. 1. Trelgol Publishing USA, 2006.

[18] T. E. Oliphant, "Python for scientific computing," *Computing in Science and Engg.*, vol. 9, pp. 10–20, May 2007.

[19] I. D. Zone, "Reference manual for intel® math kernel library 11.2," 2014.

[20] A. Jeffrey, *Matrix operations for engineers and scientists: an essential guide in linear algebra*. Springer Science & Business Media, 2010.

[21] S. Keshav, *Mathematical foundations of computer networking*. Addison-Wesley, 2012.

[22] www.mathwork.com, "Matlab."