Morgan-Stone Lattices

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ABSTRACT. *Morgan-Stone (MS) lattices* are axiomatized by the constant-free identities of those axiomatizing *Morgan-Stone (MS) algebras*. Applying the technique of characteristic functions of prime filters as homomorphisms from lattices onto the two-element chain one and their products, we prove that the variety of MS lattices is the abstract hereditary multiplicative class generated by a six-element one with an equational disjunctive system expanding the direct product of the three- and two-element chain distributive lattices, in which case subdirectly-irreducible MS lattices are exactly isomorphic copies of non-one-element subalgebras of the six-element generating MS lattice, and so we get a sixteen-element non-chain distributive lattice of varieties of MS lattices subsuming the four-/three-element chain one of "De Morgan"/Stone lattices/algebras (viz., constant-free versions of De Morgan algebras)/(more precisely, their term-wise definitionally equivalent constant-free versions, called *Stone lattices*). Among other things, we provide an REDPC scheme for MS lattices. Laying a special emphasis onto the [quasi-]equational join (viz., the [quasi-]variety generated by the union) of De Morgan and Stone laattices, we find a fifteen-element non-chain distributive lattice of its sub-quasi-varieties subsuming the eight-element one of those of the variety of De Morgan lattices found earlier, each of the rest being the quasi-equational join of its intersection with the variety of De Morgan lattices and the variety of Stone lattices.

1. INTRODUCTION

The notion of *De Morgan lattice*, being originally due to [11], has been independently explored in [7] under the term *distributive i-lattice* w.r.t. their subdirectly-irreducibles and the lattice of varieties. They satisfy so-called *De Morgan identities*. On the other hand, these are equally satisfied in *Stone algebras* (cf., e.g., [5]). This has inevitably raised the issue of unifying such varieties. Perhaps, a first way of doing it within the framework of De Morgan algebras (viz., bounded De Morgan lattices; cf., e.g., [1]) has been due to [2] (cf. [17]) under the term *Morgan-Stone (MS) algebra* providing a description of their subdirectly-irreducibles, among which there are those being neither De Morgan nor Stone algebras. Here, we study unbounded MS algebras naturally called *Morgan-Stone (MS) lattices*. Demonstrating the usefulness of the technique of the characteristic functions of prime filters and functional products of former ones as well as disjunctive systems, we briefly discuss the issues of subdirectly-irreducible Morgan-Stone lattices and their varieties. Likewise, summarizing construction of REDPC schemes (cf. [4]) for distributive lattice[expansion]s originally being due to [6] [and [8, 15]], we provide that for Morgan-Stone lattices and an enhanced one for the {quasi-}equational join of De Morgan and Stone lattices. Nevertheless, the main purpose of this study is to find the lattice of sub-quasi-varieties of the latter upon the basis of that of the variety of De Morgan lattices found in [12].

The rest of the work is as follows. Section 2 is a concise summary of basic set-theoretical and algebraic issues underlying the work. Then, in Section 3 we briefly

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summarize general issues concerning REDPC in the sense of [4] as well as equational implicative/disjunctive systems in the sense of [14]/[13] in connection with simplicity/"subdirect irreducibility". Next, Section 4 is devoted to preliminary study of Morgan-Stone lattices. Further, Section 5 is a thorough collection of culminating results on sub-quasi-varieties of the [quasi-]equational join of De Morgan and Stone lattices. Finally, Section 6 is a concise collection of open issues.

2. General background

2.1. **Set-theoretical background.** Non-negative integers are identified with the sets/ordinals of lesser ones, "their set/ordinal"|"the ordinal class" being denoted by $\omega|\infty$. Unless any confusion is possible, one-element sets are identified with their elements.

For any sets A, B and D as well as $\theta \subseteq A^2$ and $g : A^2 \to A$, let $\wp_{[K]}((B,)A)$ be the set of all subsets of A (including B) [of cardinality in $K \subseteq \infty$], $((\Delta_A|\nu_\theta)\|(A/\theta)\|\chi_A^B) \triangleq (\{\langle a, a|\theta[\{a\}]\rangle \mid a \in A\}\|\nu_\theta[A]\|(((A \cap B) \times \{1\}) \cup ((A \setminus B) \times \{0\})))$, $A^{*|+} \triangleq (\bigcup_{m \in (\omega \setminus (0|1))} A^m)$ and $g_+ : A^+ \to A, \langle[\langle a, b\rangle,]c\rangle \mapsto [g]([g_+(\langle a, b\rangle),]c)$, A-tuples {viz., functions with domain A} being written in the sequence form $\bar{t}$ with t_a, where $a \in A$, standing for $\pi_a(\bar{t})$. Then, for any $(\bar{a}|C) \in (A^*|\wp(A))$, by induction on the length (viz., domain) of any $\bar{b} = \langle[\bar{c}, d]\rangle \in A^*$, put $\underline{((\bar{a} * \bar{b})|(\bar{b}(\cap/\setminus)C))} \triangleq \underline{(([\langle]\bar{a}[*\bar{c}, d\rangle])|(\langle[\bar{c}(\cap/\setminus)C(, d)]\rangle))}$ $|[$(provided $d \in / \notin C$)$]$. Likewise, given any $\overline{S} \in \Upsilon^B$ and $\bar{f} \in \prod_{b \in B} S_b^A$, let $(\prod \bar{f}) : A \to (\prod_{b \in B} S_b), a \mapsto \langle f_b(a)\rangle_{b \in B}$, in which case

$$\ker(\prod \bar{f}) = (A^2 \cap (\bigcap_{b \in B} (\ker f_b))), \tag{2.1}$$

$$\forall b \in B : f_b = ((\prod \bar{f}) \circ \pi_b), \tag{2.2}$$

$f_0 \times f_1$ standing for $(\prod \bar{f})$, whenever $B = 2$.

A *lower/upper cone of* a poset $\mathcal{P} = \langle P, \leqq\rangle$ is any $C \subseteq P$ such that, for all $a \in C$ and $b \in P$, $(a \geqq / \leqq b) \Rightarrow (b \in C)$. Then, an $a \in S \subseteq P$ is said to be *minimal/maximal in* S, if $\{a\}$ is a lower/upper cone of S, their set being denoted by $(\min/\max)_{\mathcal{P}|\leqq}(S)$.

An $X \in Y \subseteq \wp(A)$ is said to be *[K-]meet-irreducible in* Y, [where $K \subseteq \infty$], if $\forall Z \in \wp_{[K]}(Y) : ((A \cap (\bigcap Z)) = X) \Rightarrow (X \in Z)$, their set being denoted by $\mathrm{MI}^{[K]}(Y)$.

2.2. **Algebraic background.** Unless otherwise specified, we deal with a fixed but arbitrary finitary functional signature Σ, Σ-algebras/"their carriers" being denoted by same capital Fraktur/Italic letters (with same indices, if any) "with denoting their class by A_Σ"/. Given any $\alpha \in (\infty \setminus 1)$, let $\mathrm{Tm}_\Sigma^\alpha$ be the carrier of the absolutely-free Σ-algebra $\mathfrak{Tm}_\Sigma^\alpha$, freely-generated by the set $V_\alpha \triangleq \{x_\beta\}_{\beta \in \alpha}$ of *(first α) variables*, and $\mathrm{Eq}_\Sigma^\alpha \triangleq (\mathrm{Tm}_\Sigma^\alpha)^2$, $\phi \approx /[\lessapprox \mid \gtrapprox]\psi$, where $\phi, \psi \in \mathrm{Tm}_\Sigma^\alpha$ /[and $\Sigma_+ \triangleq \{\wedge, \vee\} \subseteq \Sigma$] meaning $\langle\phi[\vee| \wedge \psi], \psi\rangle$ "and being called a Σ-*equation of rank* α"/. /[Likewise, for any Σ-algebra $\mathfrak{A}$ and $a, b \in A$, $a(\leqslant \mid \geqslant)^{\mathfrak{A}}b$ stands for $a = (a(\wedge|\vee)^{\mathfrak{A}}b)$.] Then, any $\langle\Gamma, \Phi\rangle \in (\wp_{\infty/(1[\cup\omega])}(\mathrm{Eq}_\Sigma^\alpha) \times \mathrm{Eq}_\Sigma^\alpha)$ /"with $\alpha \in \omega$" is called a Σ-*implication/-[quasi-]identity of rank* α, written as $\Gamma \to \Phi$ /[and identified with Φ] as well as treated as the universal infinitary/first-order /[positive] strict Horn sentence $\forall_{\beta \in \alpha} x_\beta((\bigwedge \Gamma) \to \Phi)$.

Subclasses of A_Σ "closed under $\mathbf{I}|\mathbf{S}|\mathbf{P}^{[\mathrm{U}]}$"/"containing each Σ-algebra with finitely-generated subalgebras in them" are referred to as *"abstract|hereditary|[ultra-]multiplicative"/local* (cf. [10]). Then, a *skeleton* {*of* a(n abstract) $\mathsf{K} \subseteq \mathsf{A}_\Sigma$} is any $\mathsf{S} \subseteq \mathsf{A}_\Sigma$ without pair-wise distinct isomorphic members {such that $\mathsf{S} \subseteq \mathsf{K} \subseteq \mathbf{IS}$ (i.e., $\mathsf{K} = \mathbf{IS}$)}. Given a $\mathsf{K} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$, set $\hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\}$

and $\mathrm{Co}_{\mathsf{K}}(\mathfrak{A}) \triangleq \{\theta \in \mathrm{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$, in which case, by the Homomorphism Theorem, we have

$$\ker[\hom(\mathfrak{A},\mathsf{K})] = \mathrm{Co}_{\mathbf{IS}\mathsf{K}}(\mathfrak{A}), \tag{2.3}$$

and so, since, for any set I, any $\overline{\mathfrak{B}} \in \mathsf{A}_\Sigma^I$ and any $\bar{f} \in (\prod_{i\in I}\hom(\mathfrak{A},\mathfrak{B}_i))$:

$$(\prod \bar{f}) \in \hom(\mathfrak{A}, \prod_{i\in I}\mathfrak{B}_i), \tag{2.4}$$

by (2.1) and (2.2) with $I \triangleq \mathrm{Co}_{\mathbf{IS}\mathsf{K}}(\mathfrak{A})$ for B, $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{B}/i\rangle_{i\in I}$ and $\bar{f} \triangleq \langle \nu_i\rangle_{i\in I}$, we get:

$$(\mathfrak{A} \in \mathbf{ISP}\mathsf{K}(=\mathbf{IP}^{\mathrm{SD}}[\mathbf{I}]\mathbf{S}_{\{>1\}}\mathsf{K})) \Leftrightarrow ((A^2 \cap (\bigcap \ker[\hom(\mathfrak{A},\mathsf{K})])) = \Delta_A). \tag{2.5}$$

According to [16], *pre-varieties* are abstract hereditary multiplicative subclasses of A_Σ (these are exactly model classes of theories constituted by Σ-implications of unlimited rank, and so are also called *implicative/implicational*; cf., e.g., [3]), $\mathbf{ISP}\mathsf{K}$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Likewise, *[quasi-]varieties* are [ultra-multiplicative] pre-varieties closed under $\mathbf{H}^{[\mathrm{I}]}[\triangleq \mathbf{I}]$ (these are exactly model classes of sets of Σ-[quasi-]identities of unlimited finite rank, and so are also called *[quasi-]equational*; cf., e.g., [10]), $\mathbf{H}^{[\mathrm{I}]}\mathbf{SP}[\mathbf{P}^{\mathrm{U}}]\mathsf{K}$ being the least one including and so called *generated by* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$. Then, intersections of a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ with [quasi-]varieties are called its *relative sub-[quasi-]varieties*, in which case, for any $\mathfrak{I} \subseteq \mathrm{Tm}_\Sigma^\omega$,

$$(\mathbf{IP}^{\mathrm{SD}}(\mathsf{K}) \cap \mathrm{Mod}(\mathfrak{I})) = \mathbf{IP}^{\mathrm{SD}}(\mathsf{K} \cap \mathrm{Mod}(\mathfrak{I})), \tag{2.6}$$

and so $\mathsf{S} \mapsto (\mathsf{S} \cap \mathsf{K})$ and $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ are inverse to one another isomorphisms between the lattices of relative sub-varieties of $\mathbf{IP}^{\mathrm{SD}}\mathsf{K}$ and those of K.

Recall that an $\mathfrak{A} \in \mathsf{A}_\Sigma$ is called *simple/[finitely-]subdirectly-irreducible*, if $\Delta_A \in (\max_\subseteq / \mathrm{MI}^{[\omega]})(\mathrm{Co}(\mathfrak{A}) \setminus (\{A^2\}/\varnothing))$, in which case $|A| \neq 1$, the class of {those of} them {which are in a(n equational) $\mathsf{K} \subseteq \mathsf{A}_\Sigma$} being denoted by $(\mathrm{Si}\,/\,\mathrm{SI}^{[\omega]})\{(\mathsf{K})\}$ {and so, by (2.3) and (2.5),

$$\mathrm{SI}(\mathbf{ISP}\mathsf{K}) \subseteq \mathbf{IS}_{>1}\mathsf{K} \tag{2.7}$$

(K being said to be *semi-simple*, if $\mathrm{SI}(\mathsf{K}) \subseteq | = \mathrm{Si}(\mathsf{K})$)}.

3. Preliminaries

A $\mho \subseteq \mathrm{Eq}_\Sigma^4$ is called an *implication scheme for* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, if this satisfies the Σ-implication:

$$(\{x_0 \approx x_1\} \cup \mho) \to (x_2 \approx x_3). \tag{3.1}$$

Likewise, it is called an *identity*|*reflexive*|*symmetric*|*transitive* one, if K satisfies the Σ-implications of the form $(\varnothing|\varnothing|\mho|(\mho \cup (\mho[x_{2+i}/x_{3+i}]_{i\in 2}))) \to \Psi$, where $\Psi \in (\mho([x_3/x_2]|[x_{2+i}/x_i]_{i\in 2}|[x_3/x_2, x_2/x_3]|[x_3/x_4]))$, reflexive symmetric transitive ones being also called *equivalence* ones. Then, $\mho$ is called a *congruence* one, if it is an equivalence one, while, for each $\varsigma \in \Sigma$ of arity $n \in (\omega \setminus 1)$, K satisfies the Σ-implications of the form $(\bigcup_{j\in n}(\mho[x_{2+i}/x_{2+i+(2\cdot j)}]_{i\in 2})) \to \Psi$, where $\Psi \in (\mho[x_{2+i}/\varsigma(\langle x_{2+i+(2\cdot j)}\rangle_{j\in n})]_{i\in 2})$.] Finally, $\mho$ [being finite] is called an *REDPC/"(equational) implicative*|*disjunctive scheme/system for* a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, if, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$, $(\forall \theta \in (\mathrm{Co}(\mathfrak{A})/\{\Delta_A\}) : (\langle a_0, a_1\rangle \in | \notin \theta) \Rightarrow (\langle a_3, a_3\rangle \in \theta)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i]_{i\in I}$ [cf. [4]/[14]|[13]] /"and so for $\mathbf{IS}[\mathbf{P}^{\mathrm{U}}]\mathsf{K}$" /"{pre-varieties generated by classes of} Σ-algebras with implicative system $\mho$ being called $\mho$*-implicative* with the class of $\langle$non-one-element$\rangle$ $\mho$-implicative members of a $\mathsf{C} \subseteq \mathsf{A}_\Sigma$ denoted by $\mathsf{C}_\mho^{\langle>1\rangle}$ {in which case, providing an $\mho$-implicative pre-variety is quasi-equational, by the Compacteness Theorem for ultra-multiplicative classes (cf., e.g., [10]), it is

$\mho'$-implicative, for some $\mho' \in \wp_\omega(\mho)$, and so the notion of implicative quasi-variety adopted here is equivalent to that adopted in [14]}"|.

3.1. **Implicativity versus REDPC and [semi-]simplicity.**

Lemma 3.1. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$ be an implication scheme for a variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$, $\mathfrak{A} \in \mathsf{V}$, $\bar{a}, \bar{b} \in A^2$ and $\theta \triangleq \theta^{\mathfrak{A}}(\bar{a})$. Suppose $\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$. Then, $\bar{b} \in \theta$.*

Proof. As (3.1) is true in $\mathsf{V} \ni (\mathfrak{A}/\theta) \models (\bigwedge \mho)[x_i/\nu_\theta(a_i), x_{2+i}/\nu_\theta(b_i)]_{i \in 2}$, while $\bar{a} \in \theta = (\ker \nu_\theta)$, we get $\bar{b} \in \theta$. □

Corollary 3.2. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$ be an implication/REDPC scheme for a variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$. Then, $\mathsf{V}^{>1}_\mho \subseteq / = \mathrm{Si}(\mathsf{V})$.*

Proof. Consider any $\mathfrak{A} \in \mathsf{V}^{>1}_\mho$ and $\vartheta \in (\mathrm{Co}(\mathfrak{A}) \setminus \{\Delta_A\})$, in which case there is some $\bar{a} \in (\vartheta \setminus \Delta_A) \neq \varnothing$, and so, for any $\bar{b} \in A^2$, $\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$. Then, "by Lemma 3.1"/ $\bar{b} \in \theta^{\mathfrak{A}}(\bar{a}) \subseteq \vartheta$, in which case $\vartheta = A^2$, and so $\mathfrak{A}$ is simple. Conversely, for any $\mathsf{A} \in \mathrm{Si}(\mathsf{V})$, $\mathrm{Co}(\mathfrak{A}) = \{\Delta_A, A^2\}$, in which case, for all $\bar{a} \in A^4$, as $\langle a_2, a_3 \rangle \in A^2$, we have $(\forall \theta \in \mathrm{Co}(\mathfrak{A}) : (a_0 \; \theta \; a_1) \Rightarrow (a_2 \; \theta \; a_3)) \Leftrightarrow ((a_0 = a_1) \Rightarrow (a_2 = a_3))$, and so $\mathfrak{A}$ is $\mho$-implicative, whenever $\mho$ is an REDPC scheme for $\mathsf{V} \ni \mathfrak{A}$. □

Theorem 3.3. *Any $\mho \subseteq \mathrm{Eq}^4_\Sigma$ is an identity congruence implication scheme for a[n equational] $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ if[f] it is an REDPC one.*

Proof. The "if" part is immediate. [Conversely, if $\mho$ is an identity congruence implication scheme for K, then, by induction on construction of any $\varphi \in \mathrm{Tm}^\omega_\Sigma$, we conclude that K satisfies the Σ-identities in $\mho[x_{2+i}/(\varphi[x_0/x_i])]_{i \in 2}$, in which case, by Mal'cev Lemma [9] (cf. [4, Lemma 2.1]), for any $\mathfrak{A} \in \mathsf{A}$, $\bar{a} \in A^2$ and $\bar{b} \in \theta^{\mathfrak{A}}(\bar{a})$, we have $\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$, and so Lemma 3.1 completes the argument]. □

Next, by Birkgoff's Theorem and (2.7), we immediately have:

Lemma 3.4. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$. Then, any variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ is $\mho$-implicative iff $\mho$ is an implicative system for* $\mathrm{SI}(\mathsf{V})$.

Likewise, as Δ_A is a congruence of any Σ-algebra $\mathfrak{A}$, by the reflexivity of implication, we equally have:

Lemma 3.5. *Any implicative system $\mho \subseteq \mathrm{Eq}^4_\Sigma$ for any $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is an identity congruence implication scheme for K.*

These lemmas, by Corollary 3.2, Theorem 3.3 and Birkgoff's one, immediately yield:

Corollary 3.6. *Let $\mho \subseteq \mathrm{Eq}^4_\Sigma$. Then, any variety $\mathsf{V} \subseteq \mathsf{A}_\Sigma$ is $\mho$-implicative iff it is semi-simple with REDPC scheme $\mho$, in which case* $(\mathrm{SI} \mid \mathrm{Si})(\mathsf{V}) = \mathsf{V}^{>1}_\mho$.

3.1.1. *Generic identity equivalence implication schemes for distributive lattice expansions.* Here, it is supposed that $\Sigma_+ \subseteq \Sigma$. Given any $\mathfrak{A} \in \mathsf{A}_\Sigma$, $X \subseteq A$ and $\Omega \subseteq \mathrm{Tm}^1_\Sigma$, we have $\Omega^{\mathfrak{A}}_X : A \to \wp(\Omega), a \mapsto \{\varphi \in \Omega \mid \varphi^{\mathfrak{A}}(a) \in X\}$.

Given any $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\, \bar{\varphi})$, $\iota \in \Omega \in \wp(V_1, \Xi)$, $i \in 2$ and $\Delta \in \wp(\Xi)$, let $\varepsilon^{i,\iota}_{\bar{\varphi},\Delta} \triangleq ((\wedge_+ \langle (\bar{\varphi} \cap \Delta) * ((\bar{\varphi} \cap \Delta) \circ [x_0/x_1]), \iota(x_{2+i}) \rangle) \lessapprox (\vee_+ \langle (\bar{\varphi} \setminus \Delta) * ((\bar{\varphi} \setminus \Delta) \circ [x_0/x_1]), \iota(x_{3-i}) \rangle)) \in \mathrm{Eq}^4_\Sigma$ and $\mho^{\bar{\varphi}}_\Omega \triangleq \{\varepsilon^{i,\iota}_{\bar{\varphi},\Delta} \mid i \in 2, \iota \in \Omega, \Delta \in \wp(\Xi)\} \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$.

Lemma 3.7. *Let $\mathfrak{A}$ be a Σ-algebra with (distributive) lattice Σ_+-reduct, $\bar{\varphi} \in (\mathrm{Tm}^1_\Sigma)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\, \bar{\varphi})$ and $\Omega \in \wp(V_1, \Xi)$. Then, $\mho^{\bar{\varphi}}_\Omega$ is an identity reflexive symmetric (transitive implication) scheme for $\mathfrak{A}$.*

Proof. Clearly, for all $j \in 2$, $\iota \in \Xi$ and $\Delta \in \wp(\Xi)$, there are some $\phi, \psi, \xi \in \mathrm{Tm}_\Sigma^3$ such that $(\varepsilon^{j,\iota}_{\bar\varphi,\Delta}[x_3/x_2]) = ((\phi \wedge \xi) \lessapprox (\psi \vee \xi))$, in which case this is satisfied in lattice Σ-expansions, and so in $\mathfrak{A}$. Likewise, there are then some $\bar\eta, \bar\zeta \in (\mathrm{Tm}_\Sigma^2)^+$ with $((\mathrm{img}\,\bar\eta) \cap (\mathrm{img}\,\bar\zeta)) \neq \varnothing$ such that $(\varepsilon^{j,\iota}_{\bar\varphi,\Delta}[x_{2+i}/x_i]_{i \in 2}) = ((\wedge_+ \bar\eta) \lessapprox (\vee_+ \bar\zeta))$, in which case this is satisfied in lattice Σ-expansions, and so in $\mathfrak{A}$. Furthermore, $(\mho^{\bar\varphi}_\Omega[x_2/x_3, x_3/x_2]) = \mho^{\bar\varphi}_\Omega$. (Next, since the Σ_+-quasi-identity $\{(x_0 \wedge x_1) \lessapprox (x_2 \vee x_3), (x_0 \wedge x_3) \lessapprox (x_2 \vee x_4)\} \to ((x_0 \wedge x_1) \lessapprox (x_2 \vee x_4))$, being satisfied in distributive latices, is so in $\mathfrak{A}$, so are logical consequences of its substitutional Σ-instances $(\mho^{\bar\varphi}_\Omega \cup (\mho^{\bar\varphi}_\Omega[x_{2+i}/x_{3+i}]_{i \in 2})) \to \Psi$, where $\Psi \in (\mho^{\bar\varphi}_\Omega[x_3/x_4])$. Finally, consider any $a \in A$ and $\bar b \in (A^2 \setminus \Delta_A)$, in which case, by the Prime Ideal Theorem, there are some $k \in 2$ and some prime filter F of $\mathfrak{A}$ such that $b_k \in F \not\ni b_{1-k}$, and so, as $\Delta \triangleq \Xi^{\mathfrak{A}}_F(a) \in \wp(\Xi)$ and $x_0 \in \Omega$, $\mathfrak{A} \not\models (\bigwedge \mho^{\bar\varphi}_\Omega)[x_i/a, x_{2+i}/b_i]_{i \in 2}$, for $\mathfrak{A} \not\models \varepsilon^{k,x_0}_{\bar\varphi,\Delta}[x_i/a, x_{2+i}/b_i]_{i \in 2}$.) ☐

This, by Corollary 3.2, immediately yields:

Corollary 3.8. *Let $\mathfrak{A}$ be a non-one-element Σ-algebra with distributive lattice Σ_+-reduct, $\bar\varphi \in (\mathrm{Tm}_\Sigma^1)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\,\bar\varphi)$ and $\Omega \in \wp(V_1, \Xi)$. Suppose $\mho^{\bar\varphi}_\Omega$ is an implicative system for $\mathfrak{A}$. Then, $\mathfrak{A}$ is simple.*

3.1.1.1. Equality determinants versus implicativity. Recall that a *(logical) Σ-matrix* is any pair $\mathcal{A} = \langle \mathfrak{A}, D \rangle$ with a Σ-algebra $\mathfrak{A}$ and a $D \subseteq A$, in which case an $\Omega \subseteq \mathrm{Tm}_\Sigma^1$ is called an *equality/identity determinant for $\mathcal{A}$*, if $\Omega^{\mathfrak{A}}_D$ is injective (cf. [13]), and so one *for* a class M of Σ-matrices, if it is so for each member of M.

Theorem 3.9. *Let M be a class of Σ-matrices and $\bar\varphi \in (\mathrm{Tm}_\Sigma^1)^*$ with $x_0 \in \Xi \triangleq (\mathrm{img}\,\bar\varphi)$. Suppose, for all $\mathcal{A} \in \mathsf{M}$, $\pi_0(\mathcal{A}){\restriction}\Sigma_+$ is a distributive lattice with set of its prime filters $\pi_1[\mathsf{M} \cap \pi_0^{-1}[\{\pi_0(\mathcal{A})\}]]$. Then, Ξ is an equality determinant for M iff $\mho^{\bar\varphi}_{V_1}$ is an implicative system for $(\mathbf{IS}_{[>1]}\{\mathbf{P}^{\mathrm{U}}\})\pi_0[\mathsf{M}]$ ([in which case its members are simple]).*

Proof. Let $\mathcal{A} = \langle \mathfrak{A}, D \rangle \in \mathsf{M}$, $\bar a \in A^2$ and, for any $\bar b \in A^2$, $h_{\bar b} \triangleq [x_i/a_i, x_{2+i}/b_i]_{i \in 2}$. First, assume Ξ is an equality determinant for M. Consider any $\bar b \in A^2$. Assume $\mathfrak{A} \not\models \varepsilon^{j,x_0}_{\bar\varphi,\Delta}[h_{\bar b}]$, for some $j \in 2$ and $\Delta \subseteq \Xi$, in which case, by the Prime Ideal Theorem, $\exists \mathcal{B} = \langle \mathfrak{A}, D' \rangle \in \mathsf{M} : \forall k \in 2 : \Delta = \Xi^{\mathfrak{A}}_{D'}(a_k)$, and so $a_0 = a_1$. Then, by Lemma 3.7 with $\Omega = \Xi$, $\mho^{\bar\varphi}_{V_1}$ is an implicative system for $\mathfrak{A}$. Conversely, assume $\mho^{\bar\varphi}_{V_1 r}$ is an implicative system for $\mathfrak{A}$ and $\Delta \triangleq \Xi^{\mathfrak{A}}_D(a_0) = \Xi^{\mathfrak{A}}_D(a_1)$. Take any $\bar b \in (D \times (A \setminus D)) \neq \varnothing$, in which case, as $\Delta \subseteq \Xi \ni x_0$, $\mathfrak{A} \not\models \varepsilon^{0,x_0}_{\bar\varphi,\Delta}[h_{\bar b}]$, for D is a prime filter of $\mathfrak{A}{\restriction}\Sigma_+$, and so $a_0 = a_1$. (Finally, Corollary 3.8 completes the argument.) ☐

3.2. **Disjunctivity.**

3.2.1. *Disjunctivity versus finite subdirect irreducibility.*

Lemma 3.10. *Any [finite] non-one-element $\mathfrak{A} \in \mathsf{A}_\Sigma$ with a disjunctive system $\mho \subseteq \mathrm{Eq}_\Sigma^4$ is finitely subdirectly-irreducible [and so subdirectly-irreducible].*

Proof. Consider any $\theta, \vartheta \in (\mathrm{Co}(\mathfrak{A}) \setminus \{\Delta_A\})$ and take any $(\bar a|\bar b) \in ((\theta|\vartheta) \setminus \{\Delta_A\}) \neq \varnothing$, in which case the Σ-identities in $\mho[x_{1|3}/x_{0|2}]$, being true in $\mathfrak{A}$, are so in $\mathfrak{A}/(\theta|\vartheta)$ (in particular, under $[x_{0|2}/\nu_{\theta|\vartheta}((a|b)_0), x_{(2|0)+i}/\nu_{\theta|\vartheta}((b|a)_i)]_{i \in 2}$), and so $\Delta_A \not\supseteq \{\langle \phi^{\mathfrak{A}}[x_i/a_i, x_{2+i}/b_i]_{i \in 2}, \phi^{\mathfrak{A}}[x_i/a_i, x_{2+i}/b_i]_{i \in 2} \rangle \mid (\phi \approx \psi) \in \mho\} \subseteq (\theta \cap \vartheta)$. Then, $(\theta \cap \vartheta) \neq \Delta_A$. Thus, induction on the cardinality of finite subsets of $\mathrm{Co}(\mathfrak{A})$ ends the proof. ☐

3.2.2. *Disjunctivity versus distributivity of lattices of sub-varieties.*

Lemma 3.11. *Let* K *be a class of* Σ*-algebras with a disjunctive system* $\mho \subseteq \mathrm{Eq}_\Sigma^4$ *as well as* R *and* S *are relative sub-varieties of* K*. Then, so is* $\mathsf{R} \cap \| \cup \mathsf{S}$*. In particular, relative sub-varieties of* K *form a distributive lattice.*

Proof. Take any $\mathfrak{I}, \mathfrak{J} \subseteq \mathrm{Tm}_\Sigma^\omega$ with $(\mathsf{R}|\mathsf{S}) = (\mathsf{K} \cap \mathrm{Mod}(\mathfrak{I}|\mathfrak{J}))$, in which case $(\mathsf{R} \cap \| \cup \mathsf{S}) = (\mathsf{K} \cap \mathrm{Mod}((\mathfrak{I} \cup \mathfrak{J}) \| \bigcup\{\mho[x_i/\phi_i, x_{2+i}\psi_i]_{i\in 2} \mid (\bar{\phi}|\bar{\psi}) \in ((\mathfrak{I}|\mathfrak{J})[x_j/x_{(2\cdot j)+(0|1)}]_{j\in\omega})\}))$, and so the distributivity of unions with intersections completes the argument. □

This, by (2.7), (2.6) and Lemma 3.10, immediately yields:

Corollary 3.12. *Let* K *be a [finite] class of finite* Σ*-algebras with a disjunctive system* $\mho \subseteq \mathrm{Eq}_\Sigma^4$ *and* P *the pre-variety generated by* K*. Suppose* P *is a variety. Then,* $\mathrm{SI}(\mathsf{P}) = \mathbf{IS}_{>1}\mathsf{K}$*, in which case* $\mathsf{S} \mapsto (\mathsf{S} \cap \mathbf{S}_{\{>1\}}\mathsf{K})$ *and* $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ *are inverse to one another isomorphisms between the lattices of sub-varieties of* P *and relative ones of* $\mathbf{S}_{\{>1\}}\mathsf{K}$*, and so they are distributive [and finite].*

Likewise, by (2.7), (2.6), Corollary 3.6 (as well as [14, Remark 2.4] and Lemma 3.11), we immediately have:

Corollary 3.13. *Let* K *be a [finite] class of [finite]* Σ*-algebras with a (finite) implicative system* $\mho \subseteq \mathrm{Eq}_\Sigma^4$ *and* P *the pre-variety generated by* K*. Suppose* P *is a variety. Then,* $(\mathrm{SI}\,|\,\mathrm{Si})(\mathsf{P}) = \mathsf{P}_\mho^{>1} = \mathbf{IS}_{>1}\mathsf{K}$*, in which case* $\mathsf{S} \mapsto (\mathsf{S} \cap \mathbf{S}_{\{>1\}}\mathsf{K})$ *and* $\mathsf{R} \mapsto \mathbf{IP}^{\mathrm{SD}}\mathsf{R}$ *are inverse to one another isomorphisms between the [finite] (distributive) lattices of sub-varieties of* P *and relative ones of* $\mathbf{S}_{\{>1\}}\mathsf{K}$*.*

4. Morgan-Stone lattices versus distributive ones

From now on, we deal with the signatures $\Sigma_+^{[-]} \triangleq (\Sigma_+[\cup\{\neg\}])$, distributive lattices being supposed to be Σ_+-algebras with their variety denoted by DL and the chain distributive lattice with carrier $n \in (\omega \setminus 2)$ and the natural ordering on this denoted by $\mathfrak{D}_n$, in which case $\epsilon_2^n \triangleq \{\langle 0,0\rangle, \langle 1, n-1\rangle\}$ is an embedding of $\mathfrak{D}_2$ into $\mathfrak{D}_n$, while, for each $i \in 2$, $\epsilon_{3:i}^4 \triangleq (\chi_3^{3\setminus(2-i)} \times \chi_3^{3\setminus(1+i)})$ is an embedding of $\mathfrak{D}_3$ into $\mathfrak{D}_2^2$, First, taking the Prime Ideal Theorem, (2.5), (2.7), Corollary 3.7 into account, we immediately have the following well-known fact (cf. [6] as to REDPC for DL):

Lemma 4.1. *Let* $\mathfrak{A} \in \mathsf{DL}$ *and* $F \subseteq A$*. Suppose* F *is either a prime filter of* $\mathfrak{A}$ *or in* $\{\varnothing, A\}$*. Then,* $h \triangleq \chi_A^F \in \hom(\mathfrak{A}, \mathfrak{D}_2)$*, in which case* $\mathsf{DL} = \mathbf{IP}^{\mathrm{SD}}\mathfrak{D}_2$*, and so* DL *is the semi-simple [pre-/quasi-]variety generated by* $\mathfrak{D}_2$ *with* $(\mathrm{Si}\,|\,\mathrm{SI})(\mathsf{DL}) = \mathbf{I}\mathfrak{D}_2$ *and REDPC scheme* $\mho_{V_1}^{\langle x_0\rangle}$*.*

A *(De-)Morgan-Stone lattice* is any Σ_+^--algebra, whose Σ_+-reduct is a distributive lattice and which satisfies the Σ_+^--identities:

$$\neg(x_0 \wedge x_1) \approx (\neg x_0 \vee \neg x_1), \tag{4.1}$$

$$x_0 \lessapprox \neg\neg x_0, \tag{4.2}$$

in which case, by (4.1), it satisfies the Σ_+^--quasi-identity:

$$(x_0 \lessapprox x_1) \to (\neg x_1 \lessapprox \neg x_0), \tag{4.3}$$

and so the Σ_+^--identities:

$$\neg(x_0 \vee x_1) \approx (\neg x_0 \wedge \neg x_1), \tag{4.4}$$

$$\neg\neg\neg x_0 \approx \neg x_0, \tag{4.5}$$

their variety being denoted by $(\mathsf{D})\mathsf{MSL}$. An $a \in A$ is called {a} *[negatively-]idempotent* {*element of* an $\mathfrak{A} \in \mathsf{MSL}$}, if $\{[\neg^{\mathfrak{A}}]a\}$ forms a subalgebra of $\mathfrak{A}$, i.e.,

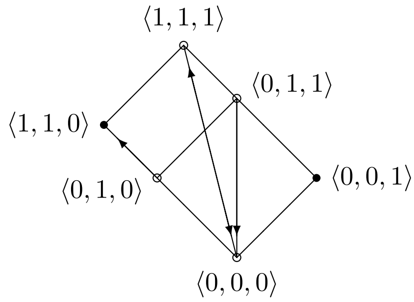

FIGURE 1. The Morgan-Stone lattice $\mathfrak{MS}_6$.

$\neg^{\mathfrak{A}}[\neg^{\mathfrak{A}}]a = [\neg^{\mathfrak{A}}]a$, with their set denoted by $\Im^{\mathfrak{A}}_{[\neg]}$, Morgan-Stone lattices with carrier of cardinality no less than $2[(-1)]$ and with[(out non-)negatively-]idempotent elements being said to be *[(totally) negatively-]idempotent*.

Let $\mathfrak{MS}_6$ be the Σ^-_+-algebra with $(\mathfrak{MS}_6{\restriction}\Sigma^-_+) \triangleq ((\mathfrak{D}^2_2{\restriction}(2^2 \setminus \{\langle 1,0\rangle\})) \times \mathfrak{D}_2)$ and $\neg^{\mathfrak{MS}_6}\bar{a} \triangleq \langle 1-a_2, 1-a_2, 1-a_1\rangle$, for all $\bar{a} \in MS_6$ (the Hasse diagram of its lattice reduct with its [non-]idempotent elements marked by [non-]solid circles and arrows reflecting action of its operation $\neg$ on its non-idempotent elements is depicted at Figure 1), in which case it is routine to check to be a Morgan-Stone lattice, and so are both $\mathfrak{MS}_5 \triangleq (\mathfrak{MS}_6{\restriction}(MS_6 \setminus \{\langle 0,0,1\rangle\}))$ and $\mathfrak{MS}_2 \triangleq (\mathfrak{MS}_5{\restriction}\{\langle i,1,0\rangle \mid i \in 2\})$ as well as, for each $j \in 2$, $\mathfrak{MS}_{4:j} \triangleq (\mathfrak{MS}_{5+j}{\restriction}(MS_{5+j} \setminus (((j+1) \times \{1\}) \times \{1-j\})))$. Likewise, let $(\mathfrak{DM}|\mathfrak{S})_{4|3}$ be the Σ^-_+-algebra with $((\mathfrak{DM}|\mathfrak{S})_{4|3}{\restriction}\Sigma^-_+) \triangleq \mathfrak{D}^{2|}_{2|3}$ and $\neg^{(\mathfrak{DM}|\mathfrak{S})_{4|3}} \triangleq ((((\pi_1{\restriction}2) \circ (2^2 \setminus \Delta_2)) \times ((\pi_0{\restriction}2) \circ (2^2 \setminus \Delta_2)))|\chi^1_3)$, in which case $\epsilon^{6|5}_{4|3} \triangleq ((((\pi_0{\restriction}2^2) \times (\pi_0{\restriction}2^2)) \times (\pi_1{\restriction}2^2))|(\epsilon^4_{3:0} \times \chi^{3\setminus 1}_3))$ is an embedding of $(\mathfrak{DM}|\mathfrak{S})_{4|3}$ into $(\mathfrak{MS}|\mathfrak{MS})_{6|5}$. Finally, for any $n \in (\{3,4\}|\{2\})$, let $(\mathfrak{K}|\mathfrak{B})_n$ be the Σ^-_+-algebra with $((\mathfrak{K}|\mathfrak{B})_n{\restriction}\Sigma^-_+) \triangleq \mathfrak{D}_n$ and $\neg^{(\mathfrak{K}|\mathfrak{B})_n} \triangleq \{\langle m, n-1-m\rangle \mid m \in n\}$, in which case $\epsilon^{3\|4}_2$ is an embedding of $\mathfrak{B}_2$ into $\mathfrak{K}_{3\|4}$, while, for every $l \in 2$, $\epsilon^4_{3:l}$ is an embedding of $\mathfrak{K}_3$ into $\mathfrak{DM}_4$, and so $\epsilon^4_{3:l} \circ \epsilon^6_4$ is that into $\mathfrak{MS}_{4:(1-l)}$. Moreover, $\{MS_6, MS_5, MS_2, \mathrm{img}(\epsilon^3_2 \circ \epsilon^5_3)\} \cup (\bigcup\{\{MS_{4:k}, \mathrm{img}(\epsilon^4_{3:k} \circ \epsilon^6_4)\} \mid k \in 2\})$ are exactly the carriers of members of $\mathbf{S}_{>1}\mathfrak{MS}_6$, in which case these are isomorphic to those of the skeleton $\mathsf{MS} \triangleq (\{\mathfrak{MS}_\ell \mid \ell \in \{6,5,2\}\} \cup \{\mathfrak{MS}_{4:\Bbbk} \mid \Bbbk \in 2\} \cup \{\mathfrak{DM}_4, \mathfrak{K}_3, \mathfrak{S}_3, \mathfrak{B}_2\})$, and so this is that of $\mathbf{IS}_{>1}\mathfrak{MS}_6$ with the embeddability partial ordering $\preceq$ between members of MS depicted at Figure 2. And what is more, $D_6 \triangleq (MS_6 \cap \pi_0^{-1}[\{1\}])$ is a prime filter of $\mathfrak{MS}_6{\restriction}\Sigma_+$, while $\Omega \triangleq \{x_0, \neg x_0, \neg\neg x_0\}$ is an equality determinant for $\langle \mathfrak{MS}_6, D_6\rangle$, in which case, by [13, Lemma 11], $\mho_\Omega \triangleq \{(\tau(x_\imath) \wedge \rho(x_{2+\jmath})) \lessapprox (\tau(x_{1-\imath}) \vee \rho(x_{3-\jmath})) \mid \imath, \jmath \in 2, \tau, \rho \in \Omega\}$ is a disjunctive system for $\mathfrak{MS}_6$, and so, for $\mathbf{IS}\mathfrak{MS}_6$. Likewise, elements of $\mathcal{PF}_4 \triangleq \{2^2 \cap \pi_o^{-1}[\{1\}] \mid o \in 2\}$ are exactly all prime filters of $\mathfrak{D}^2_2$, while $\Xi \triangleq \{x_0, \neg x_0\}$ is an equality determinant for $\mathsf{M} \triangleq (\{\mathfrak{A}\} \times \mathcal{PF}_4)$, in which case, by Theorem 3.9, $\mho^{\langle x_0, \neg x_0\rangle}_{V_1}$ is an implicative system for $\mathbf{IS}_{[>1]}\mathfrak{DM}_4$ [and so its members are simple]. On the other hand, by (4.1), (4.4), (4.5), Corollaries 3.2, 3.7 and Theorem 3.3, $\mho^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}_\Omega$ is an REDPC scheme for $\mathsf{MSL} \supseteq \mathsf{MS}$, in which case any simple member $\mathfrak{A}$ of it is $\mho^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}_\Omega$-implicative, and so all those members of MS, which are embeddable into $\mathfrak{A}$, being $\mho^{\langle x_0, \neg x_0, \neg\neg x_0\rangle}_\Omega$-implicative as well, are simple. And what is more,

$$\chi^{3\setminus 1}_3 = (\epsilon^5_3 \circ \pi_2) \in \hom(\mathfrak{S}_3, \mathfrak{B}_2), \tag{4.6}$$

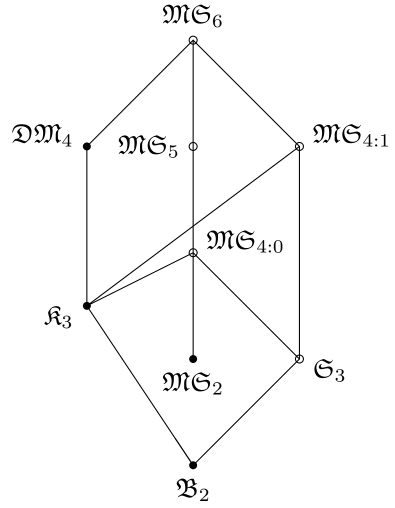

FIGURE 2. The poset $\langle \mathsf{MS}, \preceq \rangle$.

in which case $(\ker \chi_3^{3\setminus 1}) \in (\mathrm{Co}(\mathfrak{S}_3) \setminus \{\Delta_3, 3^2\})$, and so $\mathfrak{S}_3$ is not simple. In this way, (non-)simple members of MS are properly depicted by (non-)solid circles at Figure 2.

Theorem 4.2. *For any prime filter F of the Σ_+-reduct of any $\mathfrak{A} \in \mathsf{MSL}$ there is an $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ with $(\ker h) \subseteq (\ker \chi_A^F)$, in which case MSL is the [pre-/quasi-]variety generated by $\mathfrak{MS}_6$ with REDPC scheme $\mho_\Omega^{\langle x_0, \neg x_0, \neg\neg x_0 \rangle}$, and so* $\mathrm{SI}(\mathsf{MSL}) = \mathbf{I}\mathsf{MS}$.

Proof. Let $f \triangleq \chi_A^F$, $G \triangleq (\neg^{\mathfrak{A}})^{-1}[(\neg^{\mathfrak{A}})^{-1}[F]]$, $H \triangleq (A \setminus (\neg^{\mathfrak{A}})^{-1}[F])$ and $h \triangleq (f \times \chi_A^G) \times \chi_A^H)$, in which case, by (2.1) and (4.5), $(\ker f) \supseteq (((\ker f) \cap (\ker \chi_A^G)) \cap (\ker \chi_A^H)) = (\ker h) \subseteq (\neg^{\mathfrak{A}} \circ h)$, while, by (4.1) and (4.4), $G|H$ is either a prime filter of $\mathfrak{A}{\restriction}\Sigma_+$ or in $\{\varnothing, A\}$, whereas, by (4.2), $F \subseteq G$, and so, by (2.2), $\pi_0(h(a)) \leqslant \pi_1(h(a))$, for all $a \in A$. Then, by (2.4), Lemma 4.1 and the Homomorphism Theorem, h is a surjective homomorphism from $\mathfrak{A}$ onto the Σ_+^--algebra $\mathfrak{B}$ with $(\mathfrak{B}{\restriction}\Sigma_+) \triangleq (\mathfrak{D}_2^3{\restriction}h[A])$ as well as $\neg^{\mathfrak{B}} \triangleq (h^{-1} \circ \neg^{\mathfrak{A}} \circ h)$, in which case $B \subseteq MS_6$, since $\pi_0(h(a)) \leqslant \pi_1(h(a))$, for all $a \in A$, and so $\mathfrak{B} = (\mathfrak{MS}_6{\restriction}h[A])$, as, for all $a \in A$, $(\neg^{\mathfrak{A}} a \in G) \Leftrightarrow (\neg^{\mathfrak{A}} a \in F) \Leftrightarrow (a \notin H)$, in view of (4.5), as well as $(\neg^{\mathfrak{A}} a \in H) \Leftrightarrow (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \notin F) \Leftrightarrow (a \notin G)$. Hence, $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$ and $(\ker h) \subseteq (\ker f)$. Thus, the Prime Ideal Theorem, (2.5) and Corollary 3.12 complete the argument. □

This, in particular, provides an REDPC scheme for Morgan-Stone algebras [2, 17], as expansion by constants alone preserves congruences, and so a uniform insight into REDPC for Stone and De Morgan algebras, originally given by separate distinct schemes in [8, 15].

Clearly, Morgan-Stone lattices, satisfying the Σ_+^--identity:

$$\neg\neg x_0 \approx | \lessapprox x_0, \tag{4.7}$$

are nothing but *De(-‖ “ ”)Morgan lattices* in the sence of [11], their variety being denoted by DML. Likewise, Morgan-Stone lattices, satisfying the Σ_+^--identity:

$$(x_0 \wedge \neg x_0) \lessapprox x_1, \tag{4.8}$$

are exactly *Stone lattices/algebras* (cf., e.g., [5]), their variety being denoted by SL. Then, members of $\mathsf{BL} \triangleq (\mathsf{DML} \cap \mathsf{SL})$ are exactly *Boolean lattices/algebras.* Further,

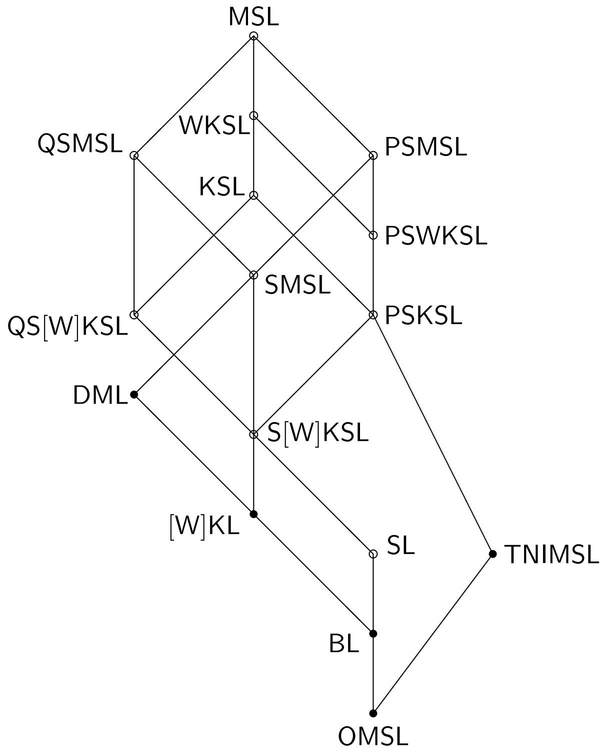

FIGURE 3. The lattice of varieties of Morgan-Stone lattices.

Morgan-Stone lattices, satisfying the former/latter of the following Σ_+^--identities:

$$x_0 \gtrapprox (\neg\neg x_0 \wedge \neg x_0), \tag{4.9}$$

$$\neg\neg x_0 \lessapprox (x_0 \vee (\neg\neg x_1 \vee \neg x_1)), \tag{4.10}$$

"in which case they satisfy the Σ_+^--quasi-identities:

$$(\neg x_0 \lessapprox x_0) \leftarrow | \rightarrow (\neg x_0 \lessapprox \neg\neg x_0), \tag{4.11}$$

in view of (4.2)"/ are said to be *quasi-/pseudo-strong*, their variety being denoted by $(\mathsf{Q/P})\mathsf{SMSL}$. Then, members of $\mathsf{SMSL} = (\mathsf{QSMSL} \cap \mathsf{PSMSL}) \supseteq (\mathsf{DML} \cup \mathsf{SL})$ are said to be *strong*. Furthermore, {*(quasi-/pseudo-)strong [weakly]*} *Kleene*{*-Stone*} *lattices* are {(quasi-/pseudo-)strong} De-Morgan{-Stone} lattices satisfying the following Σ_+^--identity:

$$(x_0 \wedge \neg x_0) \lessapprox (\neg x_1 \vee [\neg\neg]x_1), \tag{4.12}$$

their variety being denoted by $\{(\mathsf{Q/P})\mathsf{S}[\mathsf{W}]\}\mathsf{K}\{\mathsf{S}\}\mathsf{L} \supseteq (\mathsf{SL} \cup \{(\mathsf{Q/P})\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L})$, in view of (4.2). Finally, the variety of totally negatively-idempotent Morgan-Stone lattices, being relatively axiomatized by the Σ_+^--identity:

$$\neg\neg x_0 \approx \neg x_0, \tag{4.13}$$

is denoted by TNIMSL. Likewise, the variety of one-element Morgan-Stone lattices, being relatively axiomatized by the Σ_+^--identity:

$$x_0 \approx x_1, \tag{4.14}$$

is denoted by OMSL.

Corollary 4.3. *Sub-varieties of* MSL *form the non-chain distributive sixteen-element lattice, whose Hasse diagram is depicted at Figure 3, any (non-)solid*

MSL	$\{\mathfrak{MS}_6\}$
PSMSL	$\{\mathfrak{MS}_5, \mathfrak{DM}_4\}$
QSMSL	$\{\mathfrak{MS}_{4:1}, \mathfrak{DM}_4\}$
WKSL	$\{\mathfrak{MS}_5, \mathfrak{MS}_{4:1}\}$
KSL	$\{\mathfrak{MS}_{4:i} \mid i \in 2\}$
PSWKSL	$\{\mathfrak{MS}_5\}$
QS[W]KSL	$\{\mathfrak{MS}_{4:1}\}$
PSKSL	$\{\mathfrak{MS}_{4:0}\}$
SMSL	$\{\mathfrak{S}_3, \mathfrak{DM}_4\}$
DML	$\{\mathfrak{DM}_4\}$
S[W]KSL	$\{\mathfrak{S}_3, \mathfrak{K}_3\}$
[W]KL	$\{\mathfrak{K}_3\}$
SL	$\{\mathfrak{S}_3\}$
BL	$\{\mathfrak{B}_2\}$
TNIMSL	$\{\mathfrak{MS}_2\}$
OMSL	$\varnothing$

TABLE 1. Maximal subdirectly-irreducibles of varieties of Morgan-Stone lattices.

circle being marked by a (non-)semi-simple|filtral|$\{\mho^{\langle x_0,\neg x_0[,\neg\neg x_0]\rangle}_{\{x_0[,\neg x_0\langle,\neg\neg x_0\rangle]\}}$*-*$\}$*implicative variety* $\mathsf{V} \subseteq \mathsf{MSL}$ *with* $\mathsf{MS_V} \triangleq \max_{\preceq}(\mathsf{MS} \cap \mathsf{V})$ *given by Table 1, in which case* $\mathrm{SI}(\mathsf{V}) = \mathbf{IS}_{>1}\mathsf{MS_V}$*, and so* V *is the* $\lceil$*pre-/quasi-*$\rceil$*variety generated by* $\mathsf{MS_V}$*. In particular,* SMSL *is the one generated by* $\lfloor\mathrm{SI}\rfloor(\mathsf{DML} \cup \mathsf{SL})$*.*

Proof. Clearly, $\mathsf{PSWKL} \ni \mathfrak{MS}_5 \not\models (4.12)[x_0/\langle 1,1,0\rangle, x_1/\langle 0,1,1\rangle]$, $(\mathsf{MSL}|\mathsf{DML}) \ni (\mathfrak{MS}|\mathfrak{DM})_{(6|4)} \not\models [(4.12)][x_0/(\langle 1,1,0\rangle|\langle 1,0\rangle), x_1/(\langle 0,0,1\rangle|\langle 0,1\rangle)]$, $\mathfrak{MS}_{4:0} \in \mathsf{PSKSL}$, $\mathsf{QSKSL} \ni \mathfrak{MS}_{4:1} \not\models (4.10)[x_0/\langle 0,1,1\rangle, x_1/\langle 0,0,1\rangle]$, $\mathsf{SL} \ni \mathfrak{S}_3 \not\models (4.7)[x_0/1]$, $\mathsf{KL} \ni \mathfrak{K}_3 \not\models (4.8)[x_i/(1-i)]_{i\in 2}$ and $(\mathsf{BL}|\mathsf{TNIMSL}) \ni (\mathfrak{B}|\mathfrak{MS})_2 \not\models (4.13|4.9)[x_0/0]$. Then, Figure 2 confirms Table 1, in which case the intersections of MS with the sixteen sub-varieties of MSL involved are exactly all lower cones of the poset $\langle \mathsf{MS}, \preceq\rangle$, and so (2.5), (2.6), (2.7) and Theorem 4.2 complete the argument. □

It is in this sense that SMSL is the implicational/[quasi-]equational join of DML and SL. The lattice of its sub-quasi-varieties is found in the next Section.

5. Quasi-varieties of strong Morgan-Stone lattices

Given any $\mathsf{K} \subseteq \mathsf{MSL}$, $[\mathsf{N}]\mathsf{IK}$ stands for the class of [non-]idempotent members of K [in which case it is the relative sub-quasi-variety of K, relatively axiomatized by the Σ^-_+-quasi-identity:

$$(\neg x_0 \approx x_0) \to (x_0 \approx x_1), \tag{5.1}$$

and so a quasi-variety, whenever K is so].

Lemma 5.1. *Any (non-one-element finitely-generated)* $\mathfrak{A} \in \mathsf{MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A}, \mathfrak{B}_2) \neq \varnothing$*, in which case* $(\mathsf{SMSL} \setminus \mathsf{NISMSL}) \subseteq \mathsf{DML}$*, and so* $\mathsf{SMSL} = (\mathsf{NISMSL} \cup \mathsf{DML})$*. In particular,* $\mathsf{NIMS} = \{\mathfrak{S}_3, \mathfrak{B}_2\}$*.*

Proof. The "if" part is by the fact that $\mathfrak{B}_2$ has no idempotent element. (Conversely, assume $\hom(\mathfrak{A}, \mathfrak{B}_2) = \varnothing$, in which case, by (4.6), $\hom(\mathfrak{A}, \mathfrak{S}_3) = \varnothing$, and so, for any $h \in \hom(\mathfrak{A}, \mathfrak{MS}_6)$, $(\mathrm{img}\, h) \nsubseteq (\mathrm{img}\, \epsilon^5_3)$, for, otherwise, we would have $(h \circ (\epsilon^5_3)^{-1}) \in \hom(\mathfrak{A}, \mathfrak{S}_3) = \varnothing$. Take any $\bar{a} \in A^+$ such that $\mathfrak{A}$ is generated by $\mathrm{img}\, \bar{a}$. Let $n \triangleq (\mathrm{dom}\, \bar{a}) \in (\omega \setminus 1)$ and $\bar{b} \triangleq \langle \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a_j \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a_j\rangle_{j\in n}$, in which case there is some $i \in n$ such that $h(a_i) \notin (\mathrm{img}\, \epsilon^5_3)$, and so $h(b_i) \in \{\langle m, m, 1-m\rangle \mid m \in 2\}$. Put [by induction on any $k \in n]$ $c_{1[+k]} \triangleq ((b_{0[+k]}[\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}c_k])[\wedge^{\mathfrak{A}}c_k])$, in which case $h(c_{1[+k]})$ is in $\{\langle\imath,\imath,\jmath\rangle \mid \langle\imath,\jmath\rangle \in (2^2 \setminus \langle 0,0\rangle)\}$, for $h(b_{0[+k]})$ is so, and so, by induction on any $l \in ((n+1)\setminus(i+1)) \ni n$, we see that $h(c_l)$ is in $\{\langle m,m,1-m\rangle \mid m \in 2\}$, for $h(b_i)$ is so. Then, $h(\neg^{\mathfrak{A}}c_n) = h(c_n)$, in which case, by (2.5) and Theorem 4.2, $\neg^{\mathfrak{A}}c_n = c_n$, and so $\mathfrak{A}$, being non-one-element, is idempotent.) Finally, (2.5), (4.6) and Corollary 4.3 complete the argument. □

This, by (2.5), Corollary 4.3, (2.1), (2.4) with $I = 2$ and the locality of quasi-varieties, immediately yields:

Corollary 5.2. *For any variety* $\mathsf{V} \subseteq \mathsf{MSL}$ $\{$*such that either* $(\mathsf{S}|\mathsf{B})\mathsf{L} \subseteq \mathsf{V}\}$*,* NIV *is the pre-/quasi-variety generated by* $\varnothing\{\cup\{\mathfrak{A} \times \mathfrak{B}_2 \mid \mathfrak{A} \in (\mathsf{MS}_\mathsf{V} \setminus \{[(\mathfrak{S}|\mathfrak{B})_{3|2}\})\} \cup (\mathsf{MS}_\mathsf{V} \cap \{(\mathfrak{S}|\mathfrak{B})_{3|2}\})\}$*, in which case* NIMSL *is the one generated by* $\{\mathfrak{MS}_6 \times \mathfrak{B}_2\}$*, while* $\mathsf{NI}\langle\mathsf{S}\rangle(\mathsf{DM}|\mathsf{K})\langle\mathsf{S}\rangle\mathsf{L}$ *is the one generated by* $\{(\mathfrak{DM}\|\mathfrak{K})_{4\|3} \times \mathfrak{B}_2\langle,\mathfrak{S}_3\rangle\}$*, whereas* $\mathsf{NI}(\mathsf{TNI} \wr \mathsf{O})\mathsf{MSL} = \mathsf{OMSL}$*, and so any (non-one-element)* $\mathfrak{A} \in \mathsf{MSL}$ *is non-idempotent if(f)* $\hom(\mathfrak{A},\mathfrak{B}_2) \neq \varnothing$*.*

Likewise, Lemma 5.1 and [12, Proof of Lemma 4.9] immediately yield:

Corollary 5.3. $\mathfrak{K}_3$ *is embeddable into any member of* $\mathsf{SKSL} \setminus \mathsf{NISKSL}$*.*

Corollary 5.4. $\mathsf{NIMSL} \cup \mathsf{TNIMSL}$ *is the sub-quasi-variety of* NIMSL *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_0 \approx x_0) \to (x_0 \approx \neg x_1) \tag{5.2}$$

and is the pre-/quasi-variety generated by $\{\mathfrak{MS}_6 \times \mathfrak{B}_2, \mathfrak{MS}_2\}$*.*

Proof. Clearly, (5.2) = (5.1$[x_1/\neg x_1]$) is true in both NIMSL and $\mathfrak{MS}_2$. Conversely, any $\mathfrak{A} \in \mathsf{IMSL}$ satisfying (5.2), has an idempotent element a, in which case, for any $b \in A$, as $\mathfrak{A} \models (5.2)[x_0/a, x_1/[\neg^{\mathfrak{A}}]b]$, we have $\neg^{\mathfrak{A}}b = a[= \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b]$, and so $\mathfrak{A} \in \mathsf{TNIMSL}$. Then, Corollaries 4.3 and 5.2 complete the argument. □

Likewise, we have:

Corollary 5.5. *For any variety* $\mathsf{V} \subseteq \mathsf{MSL}$ *such that* $\mathsf{V} \not\subseteq [\mathsf{W}]\mathsf{KSL}$*, the class* $\mathsf{NIV} \cup (\mathsf{V} \cap [\mathsf{W}]\mathsf{KSL})$ *is the sub-quasi-variety of* V *relatively axiomatized by the* Σ^-_+*-quasi-identity:*

$$(\neg x_0 \approx x_0) \to (x_0 \lessapprox ([\neg\neg]x_1 \vee \neg x_1)) \tag{5.3}$$

and is the pre-/quasi-variety generated by $\mathsf{MS}_{\mathsf{V}\cap[\mathsf{W}]\mathsf{KSL}} \cup \{\mathfrak{A} \times \mathfrak{B}_2 \mid \mathfrak{A} \in (\mathsf{MS}_\mathsf{V} \setminus \{\mathfrak{S}_3, \mathfrak{B}_2\})\}$*. In particular,* $\mathsf{NI}\{\mathsf{S}\}\mathsf{DM}\{\mathsf{S}\}\mathsf{L} \cup \{\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L}$ *is the sub-quasi-variety of* $\{\mathsf{S}\}\mathsf{DM}\{\mathsf{S}\}\mathsf{L}$ *relatively axiomatized by either of* (5.3) *and is the pre-/quasi-variety generated by* $\{\mathfrak{DM}_4 \times \mathfrak{B}_2, \mathfrak{K}_3\{,\mathfrak{S}_3\}\}$*.*

Proof. Clearly, (5.3) is satisfied in $\mathsf{NIV} \cup (\mathsf{V} \cap [\mathsf{W}]\mathsf{KSL})$. Conversely, consider any $\mathfrak{A} \in \mathsf{IV}$ satisfying (5.3) and any $a, b \in A$, in which case there is some $c \in A$ such that $\neg^{\mathfrak{A}}c = c$, and so, as $\mathfrak{A}(5.3)[x_0/c, x_1/(a|b)]$, we have $c \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}}(a|b) \vee^{\mathfrak{A}} [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}](a|b))$. Then, by (4.2), (4.3) and (4.4) [as well as (4.5)], we get $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a) \leqslant^{\mathfrak{A}} c$, in which case $\mathfrak{A} \in (\mathsf{V} \cap [\mathsf{W}]\mathsf{KSL})$, and so Corollaries 4.3 and 5.2 complete the argument. □

This, by Lemma 5.1 and [12, Case 8 of Proof of Theorem 4.8], immediately yields:

Corollary 5.6. $\mathfrak{DM}_4$ *is embeddable into any member of* $\{\mathsf{S}\}\mathsf{DM}\{\mathsf{S}\}\mathsf{L}$ *not satisfying* (5.3)*.*

Members of $\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}\lfloor\mathsf{W}\rfloor\mathsf{K}\{\mathsf{S}\}\mathsf{L}$, satisfying the Σ_{+}^{-}-quasi-identity:

$$\{\neg x_0 \lessapprox x_0, (x_0 \wedge \neg x_1) \lessapprox (\neg x_0 \vee x_1)\} \to (\neg x_1 \lessapprox [\neg\neg]x_1), \tag{5.4}$$

are called *[weakly-]regular*, their quasi-variety being denoted by

$$[\mathsf{W}]\mathsf{R}\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}\lfloor\mathsf{W}\rfloor\mathsf{K}\{\mathsf{S}\}\mathsf{L}$$
$$[= \{\langle\lceil| \supseteq\rceil\rangle\}(\mathsf{R}\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}\lfloor\mathsf{W}\rfloor\mathsf{K}\{\mathsf{S}\} \cup (\mathsf{OMSL}\{\langle\lceil|\mathsf{TNIMSL}\rceil\rangle\}))$$

in view of $(4.11)\{\langle\lceil|(4.2)\rceil\rangle\}]$.

Given any Morgan-Stone lattice $\mathfrak{A}(\in [\mathsf{W}]\{\langle\lceil\mathsf{Q}\|\mathsf{P}\rceil\mathsf{S}\rangle\}[\mathsf{W}]\mathsf{K}\{\mathsf{S}\}\mathsf{L})$, by (4.1), (4.3) and (4.4) (as well as [(4.2) and] (4.12)), $(\mathfrak{I}|\mathfrak{F})^{\mathfrak{A}}_{[\mathrm{W}]} \triangleq \{a \in A \mid [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]a(\leqslant \mid \geqslant)^{\mathfrak{A}}\neg^{\mathfrak{A}}a\} \supseteq \{b(\wedge|\vee)^{\mathfrak{A}}\neg^{\mathfrak{A}}b \mid b \in A\} \neq \varnothing$, for $A \neq \varnothing$, is (an|a ideal|filter of $\mathfrak{A}{\upharpoonright}\Sigma_+$) such that $\neg^{\mathfrak{A}}[(\mathfrak{I}|\mathfrak{F})^{\mathfrak{A}}_{[\mathrm{W}]}] \subseteq (\mathfrak{F}|\mathfrak{I})^{\mathfrak{A}}_{[\mathrm{W}]}$ (in which case $\mathfrak{R}^{\mathfrak{A}}_{[\mathrm{W}]} \triangleq ((\mathfrak{F}^{\mathfrak{A}}_{[\mathrm{W}]} \times \{1\}) \cup (\mathfrak{I}^{\mathfrak{A}}_{[\mathrm{W}]} \times \{0\}))$ forms a subalgebra of $\mathfrak{A} \times \mathfrak{B}_2$ such that, for every $\bar{d} \in \mathfrak{R}^{\mathfrak{A}}_{[\mathrm{W}]}$, $(d_1 = 1) \Rightarrow (d_0 \in \mathfrak{F}^{\mathfrak{A}}_{[\mathrm{W}]})$, and so, by Corollary 4.3, the *[weak] regularization* $\mathfrak{R}_{[\mathrm{W}]}(\mathfrak{A}) \triangleq ((\mathfrak{A} \times \mathfrak{B}_2){\upharpoonright}\mathfrak{R}^{\mathfrak{A}}_{[\mathrm{W}]})$ *of* $\mathfrak{A}$ is in $[\mathsf{W}]\mathsf{R}\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}[\mathsf{W}]\mathsf{K}\{\mathsf{S}\}\mathsf{L}$. Then, $(\pi_0{\upharpoonright}\mathfrak{R}^{\mathfrak{S}_3}) \in \hom(\mathfrak{R}(\mathfrak{S}_3), \mathfrak{S}_3)$ is bijective, so, by Corollary 4.3, $\mathfrak{S}_3 \in \mathsf{RSKSL}$. Likewise, $(\epsilon^4_2\|\{\langle i, \langle\chi_4^{4\setminus 3}(i) + \chi_4^{4\setminus 1}(i), \chi_4^{4\setminus 2}(i)\rangle\rangle \mid i \in 4\}) \in \hom((\mathfrak{B}\|\mathfrak{K})_{2\|4}, \mathfrak{K}_4\|\mathfrak{R}(\mathfrak{K}_3))$ is injective‖bijective, so, by Corollary 4.3, $(\mathfrak{B}\|\mathfrak{K})_{2\|4} \in \mathsf{RKL}$.)

Lemma 5.7. $[\mathsf{W}]\mathsf{R}\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}\lfloor\mathsf{W}\rfloor\mathsf{K}\{\mathsf{S}\}\mathsf{L} \subseteq (\mathsf{NI}\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}\lfloor\mathsf{W}\rfloor\mathsf{K}\{\mathsf{S}\}\mathsf{L}[\cup\mathsf{TNIMSL}])$.

Proof. Consider any $\mathfrak{A} \in [\mathsf{W}]\mathsf{R}\{\langle\lceil\mathsf{Q}|\mathsf{P}\rceil\mathsf{S}\rangle\}\lfloor\mathsf{W}\rfloor\mathsf{K}\{\mathsf{S}\}\mathsf{L}$ and any $a, b \in A$ such that $\neg^{\mathfrak{A}}a = a$, in which case, as $\mathfrak{A} \models (4.1\|5.4)[x_0/a, x_1/((\neg^{\mathfrak{A}})b|(a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}})b))]$ [and $\mathfrak{A} \models (4.4)[x_0/\neg^{\mathfrak{A}}a, x_1/(\neg^{\mathfrak{A}})\neg^{\mathfrak{A}}b)]$ (as well as $\mathfrak{A} \models (4.2[\wr 4.5])[x_0/b]$), we have $(b \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}})\neg^{\mathfrak{A}}b \leqslant^{\mathfrak{A}} (\neg^{\mathfrak{A}}a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})b) = \neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}})b) \leqslant^{\mathfrak{A}} [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}](a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}})b) = (a \wedge^{\mathfrak{A}} [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}](\neg^{\mathfrak{A}})b) \leqslant^{\mathfrak{A}} [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}](\neg^{\mathfrak{A}})b(= \neg^{\mathfrak{A}}b)$, and so [by (4.3)] $b \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}b \leqslant^{\mathfrak{A}} [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]b[\leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}b]$, i.e., $\neg^{\mathfrak{A}}b = [\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]b$. Then, since $\mathfrak{A} \models (4.12)[x_0/(a|[\neg^{\mathfrak{A}}]b), x_1/(b|a)]$, we have $[\neg^{\mathfrak{A}}]b \leqslant^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} [\neg^{\mathfrak{A}}]b$, i.e., $a = [\neg^{\mathfrak{A}}]b$, and so [by Corollary 5.4] $\mathfrak{A}$ is [either] non-idempotent [or totally negatively-idempotent]. □

Corollary 5.8. *$\mathfrak{K}_4$ is embeddable into any $\mathfrak{A} \in (\mathsf{NIQSMSL} \setminus \mathsf{SL}) \supseteq (\mathsf{RQSKSL} \setminus \mathsf{SL})$.*

Proof. Then, there are some $a, b \in A$ such that $c \triangleq (a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a) \neq d \triangleq (b \wedge^{\mathfrak{A}} c) \leqslant^{\mathfrak{A}} c$, in which case, applying (4.1) and (4.3) [twice], we have $[\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}d \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]c \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}c \leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}}d$, and so, by (4.2) and (4.9), we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d) = (c|d)$. In this way, as $c \neq d$, by (5.1), we have $\neg^{\mathfrak{A}}c \neq c$, in which case we get $\neg^{\mathfrak{A}}d \neq \neg^{\mathfrak{A}}c$, and so $\{\langle 0, d\rangle, \langle 1, c\rangle, \langle 2, \neg^{\mathfrak{A}}c\rangle, \langle 3, \neg^{\mathfrak{A}}d\rangle\}$ is an embedding of $\mathfrak{K}_4$ into $\mathfrak{A}$. Finally, Lemma 5.7 completes the argument. □

Theorem 5.9. *Let $\mathsf{V} \triangleq \{(\langle\mathsf{Q}[\|\mathsf{P}]\rangle\mathsf{S})\}[\mathsf{W}]\mathsf{K}\{\mathsf{S}\}\mathsf{L}$ and $\mathsf{K} \triangleq (\varnothing\{\cup(\mathsf{MS}_{\mathsf{V}} \cap (\{\mathfrak{S}_3\}(\cup\langle\varnothing[\|\{\mathfrak{M}\mathfrak{S}_2\}]\rangle))))\})$. Then, $\mathsf{QV} \triangleq [\mathsf{W}]\mathsf{R}\{(\langle\mathsf{Q}[\|\mathsf{P}]\rangle\mathsf{S})\}[\mathsf{W}]\mathsf{K}\{\mathsf{S}\}\mathsf{L}$ is the pre-/quasi-variety generated by $\mathfrak{R}_{[\mathrm{W}]}[\mathsf{MS}_{\mathsf{V}} \setminus \mathsf{K}] \cup \mathsf{K}$. In particular, $\mathsf{R}\{\mathsf{S}\}\mathsf{K}\{\mathsf{S}\}\mathsf{L}$ is the one generated by $\{\mathfrak{K}_4\{, \mathfrak{S}_3\}\}$.*

Proof. Consider any finitely-generated $\mathfrak{A} \in (\mathsf{Q} \setminus (\mathsf{OMSL}[\cup\mathsf{TNIMSL}]))$. Take any $\bar{a} \in A^+$ such that $\mathfrak{A}$ is generated by $\operatorname{img}\bar{a}$. Let $n \triangleq (\operatorname{dom}\bar{a}) \in (\omega \setminus 1)$ and $b \triangleq (\wedge^{\mathfrak{A}}_{+}\langle\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}a_m \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}a_m\rangle_{m\in n})$, in which case, by (4.1), (4.4) and (4.12), we have $\neg^{\mathfrak{A}}b \leqslant^{\mathfrak{A}} b$. Consider any $h \in \hom(\mathfrak{A}, \mathfrak{M}\mathfrak{S}_6)$. Let $(I|J) \triangleq \{i \in n \mid h(a_i) \notin (F|I)^{\mathfrak{S}_6}_{[\mathrm{W}]}\}$, $(\imath|\jmath) = |(I|J)|$ and $\bar{\Bbbk}|\bar{\ell}$ any bijection from $\imath|\jmath$ onto $I|J$. We prove, by contradiction, that there is some $g \in \hom(\mathfrak{A}, \mathfrak{B}_2)$ such that $g[\operatorname{img}((\bar{\Bbbk}|\bar{\ell}) \circ \bar{a})] = \{0|1\}$. For suppose that, for every $g \in \hom(\mathfrak{A}, \mathfrak{B}_2)$, there is either some $i \in \imath$ or some $j \in \jmath$ such that $g(a_{(\Bbbk|\ell)_{i|j}})) = (1|0)$, in which case, as, by Lemmas 5.1

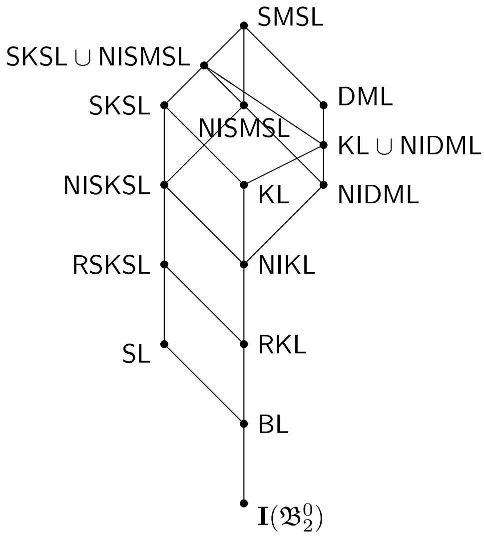

FIGURE 4. The lattice of pre-/quasi-varieties of strong Morgan-Stone lattices.

and 5.7, $\hom(\mathfrak{A},\mathfrak{B}_2)\neq\varnothing$, we have $(I\cup J)\neq\varnothing$, and so we are allowed to put $c\triangleq(\vee_+^{\mathfrak{A}}((\bar{\Bbbk}\circ\bar{a}[\circ\neg^{\mathfrak{A}}\circ\neg^{\mathfrak{A}}])*(\bar{\ell}\circ\bar{a}\circ\neg^{\mathfrak{A}})))$. Then, $\pi_{0\wr 2}(h([\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]c))=0$, in which case [by (4.5)] $\pi_0(h(\neg^{\mathfrak{A}}c))=1$, and so $\neg^{\mathfrak{A}}c\nleqslant^{\mathfrak{A}}[\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}]c$, for $(h\circ\pi_0)\in\hom(\mathfrak{A}{\restriction}\Sigma_+,\mathfrak{D}_2)$. Now, consider any $f\in\hom(\mathfrak{A},\mathfrak{MS}_6)$, in which case $(\mathfrak{MS}_6{\restriction}(\operatorname{img}f))\in\mathsf{V}\not\ni\mathfrak{DM}_4$, in view of Corollary 4.3, and so $(\operatorname{img}\epsilon_4^6)\nsubseteq(\operatorname{img}f)$, i.e., $\Im^{\mathcal{MS}_6}=\epsilon_4^6[2^2\setminus\Delta_2]\nsubseteq(\operatorname{img}f)$. Consider the following complementary cases:

- $(\operatorname{img}f)\subseteq(\operatorname{img}\epsilon_3^5)$,
  in which case, by (4.6), $e\triangleq(f\circ(\epsilon_3^5)^{-1}\circ\chi_3^{3\setminus 2})\in\hom(\mathfrak{A},\mathfrak{B}_2)$, and so, by the assumption to be disproved, $\pi_{1\wr 2}(f(c))=e(c)=1$. Then, $f(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c)=\langle 0,0,0\rangle\leqslant^{\mathfrak{MS}_6}f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$.
- $(\operatorname{img}f)\nsubseteq(\operatorname{img}\epsilon_3^5)$,
  in which case there is some $m\in n$ such that $f(a_m)\notin(\operatorname{img}\epsilon_3^5)\not\supseteq\Im^{\mathfrak{MS}_6}$, in which case $f(b)\in\Im^{\mathfrak{MS}_6}$, and so $f(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c)\leqslant^{\mathfrak{MS}_6}f(b)=f(\neg^{\mathfrak{A}}b)\leqslant^{\mathfrak{MS}_6}f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$.

Thus, anyway, $f(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c)\leqslant^{\mathfrak{MS}_6}f(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$, in which case, by (2.5) and Theorem 4.2, $(b\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}c)\leqslant^{\mathfrak{A}}(\neg^{\mathfrak{A}}b\vee^{\mathfrak{A}}c)$, and so $\mathfrak{A}\not\models(5.4)[x_0/b,x_1/c]$. This contradiction to the [weak] regularity of $\mathfrak{A}$ definitely shows that, for each $\mathfrak{B}\in\mathsf{MS_V}\subseteq\mathbf{IS}\mathfrak{MS}_6$ and every $h'\in\hom(\mathfrak{A},\mathfrak{B})$, there is some $g'\in\hom(\mathfrak{A},\mathfrak{B}_2)$ such that $(\operatorname{img}f')\subseteq\Re_{[\mathrm{W}]}^{\mathfrak{B}}$, where $f'\triangleq(h'\times g')$, in which case, by (2.4), $f'\in\hom(\mathfrak{A},\Re_{[\mathrm{W}]}(\mathfrak{B}))$, while, by (2.1), $(\ker f')\subseteq(\ker h')$, and so the locality of quasi-varieties, (2.5) and Corollary 4.3 complete the argument. □

Thus, the apparatus of regularizations of Kleene-Stone lattices involved here yields a more transparent and immediate insight/proof into/to [14, Proposition 4.7].

Lemma 5.10. *$\mathfrak{K}_3\times\mathfrak{B}_2$ is embeddable into any $\mathfrak{A}\in(\mathsf{NISKSL}\setminus\mathsf{RSKSL})$.*

Proof. Then, by (4.1), (4.3), (4.4) and (4.5), there are some $a,b\in A$ such that $(c|d)\triangleq\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a|b)(\geqslant|\ngeqslant)^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$ and $(c\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}d)\leqslant^{\mathfrak{A}}(\neg^{\mathfrak{A}}c\vee^{\mathfrak{A}}d)$, in which case, using (4.1), (4.4) and (4.5), by induction on construction of any $\varphi\in\mathrm{Tm}^2_{\Sigma_+^-}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c,d)=\varphi^{\mathfrak{A}}(c,d)$, and so the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c,d\}$ is a non-idempotent Kleene lattice such that $\mathfrak{B} \not\models (5.4)[x_0/c, x_1/d]$. Hence, $\mathfrak{K}_3 \times \mathfrak{B}_2$ being embeddable into $\mathfrak{B}$, by [12, Case 4 of Proof of Theorem 4.8], is so into $\mathfrak{A}$. □

Lemma 5.11. *$\mathfrak{DM}_4 \times \mathfrak{B}_2$ is embeddable into any $\mathfrak{A} \in (\mathsf{NISMSL} \setminus \mathsf{SKSL})$.*

Proof. Then, there are some $a, b \in A$ such that, by (4.2), $c \triangleq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \nleqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} b)$, in which case, by (4.1), (4.4) and (4.5), we have both $\neg^{\mathfrak{A}}(c|d)(\geqslant | \leqslant)^{\mathfrak{A}}(c|d) = \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}(c|d)$, and so, by induction on construction of any $\varphi \in \mathrm{Tm}^2_{\Sigma^-_+}$, we get $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$. Thus, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{c, d\}$ is a non-idempotent De Morgan lattice such that $\mathfrak{B} \not\models (4.12)[x_0/c, x_1/d]$, in which case, by the proof of [12, Lemma 4.10], $\mathfrak{DM}_4 \times \mathfrak{B}_2$ is embeddable into $\mathfrak{B}$, and so into $\mathfrak{A}$. □

Lemma 5.12. *Let $\mathfrak{A} \in \mathsf{QSMSL}$ and $a \in A$. Suppose $\neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a \neq a$. Then, $b \triangleq (\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} c \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leqslant^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a)$, while both $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$ and $\neg^{\mathfrak{A}} b = d$, whereas $b \neq c \neq d$, in which case $\{\langle 0, b\rangle, \langle 1, c\rangle, \langle 2, d\rangle\}$ is an embedding of $\mathfrak{S}_3$ into $\mathfrak{A}$, and so $\mathfrak{S}_3$ is embeddable into any member of $(\mathsf{QSMSL} \setminus \mathsf{DML})$.*

Proof. In that case, by (4.2), $b \leqslant^{\mathfrak{A}} c \leqslant^{\mathfrak{A}} d$, while, by (4.1), (4.4) and (4.5), both $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$ and $\neg^{\mathfrak{A}} b = d$, whereas $c \neq d$, for, otherwise, since $\mathfrak{A} \models (4.2|4.9)[x_0/a]$, $\{b, \neg^{\mathfrak{A}} a, a, \neg^{\mathfrak{A}}\neg^{\mathfrak{A}} a, d\}$ would be a pentagon of the distributive lattice $\mathfrak{A}{\restriction}\Sigma_+$, and so $b \neq c$, for otherwise, we would have $c = b = \neg^{\mathfrak{A}} c = \neg^{\mathfrak{A}} b = d$. □

Theorem 5.13. *Sub-pre/quasi-varieties of SMSL form the fifteen-element non-chain distributive lattice depicted at Figure 4.*

Proof. We use Corollary 4.3 tacitly. Clearly, $\mathfrak{DM}_4 \times \mathfrak{B}_2$ is not in SKSL, for $\mathfrak{DM}_4$ is not so, while $\pi_0{\restriction}(2^2 \times \Delta_2)$ is a surjective homomorphism from the former onto the latter, in which case, by Corollary 5.5, $\mathsf{SKSL} \subsetneq (\mathsf{SKSL} \cup \mathsf{NISMSL}) \subsetneq \mathsf{SMSL}$, for $\mathsf{SMSL} \ni \mathfrak{DM}_4 \not\models (5.3)[x_i/\langle i, 1-i\rangle]_{i\in 2}$. Likewise, $\mathfrak{S}_3 \notin \mathsf{DML}$, so, by Corollaries 5.2, 5.5 and Theorem 5.9, $(\mathsf{KL} \cup \mathsf{NIDML}) \subsetneq (\mathsf{SKSL} \cup \mathsf{NISMSL})$, $\mathsf{NIDML} \subsetneq \mathsf{NISMSL}$, $\mathsf{NIKL} \subsetneq \mathsf{NISKSL}$ and $\mathsf{RKL} \subsetneq \mathsf{RSKSL}$, while, by Corollary 5.2, $\mathsf{NIKL} \ni (\mathfrak{K}_3 \times \mathfrak{B}_2) \not\models (5.4)[x_0/\langle\langle 0, 1\rangle, \langle 1, 1\rangle\rangle, x_1/(\langle\langle 0, 0\rangle, \langle 1, 1\rangle\rangle]$, so, by Lemma 5.7, $\mathsf{RSKSL} \subsetneq \mathsf{NISKSL}$, as well as $\mathsf{KL} \ni \mathfrak{K}_3 \not\models (5.1)[x_0/\langle 0, 1\rangle, x_1/\langle 0, 0\rangle]$, so $\mathsf{NISKSL} \subsetneq \mathsf{SKSL}$. Finally, by Theorem 5.9, $\mathfrak{S}_3 \in \mathsf{RSKSL} \ni \mathfrak{K}_4 \not\models (4.8)[x_i/(1-i)]_{i\in 2}$, so $\mathsf{SL} \subsetneq \mathsf{RSKSL}$. Thus, by Lemma 5.1, Corollaries 5.2, 5.5, Theorem 5.9 and [12, Theorem 4.8], the fifteen quasi-varieties involved are pair-wise distinct and do form the lattice depicted at Figure 4. Now, consider any pre-variety $\mathsf{P} \subseteq \mathsf{SMSL}$ such that $\mathsf{P} \nsubseteq \mathsf{DML}$, in which case, by Lemma 5.12, $\mathfrak{S}_3 \in \mathsf{P}$, and so $\mathsf{SL} \subseteq \mathsf{P}$, as well as the following exhaustive cases:

(1) $\mathsf{P} \nsubseteq (\mathsf{SKSL} \cup \mathsf{NISMSL})$,
in which case, by Corollaries 5.5 and 5.6, $\mathfrak{DM}_4 \in \mathsf{P} \ni \mathfrak{S}_3$, and so $\mathsf{P} = \mathsf{SMSL}$.
(2) $\mathsf{P} \subseteq (\mathsf{SKSL} \cup \mathsf{NISMSL})$ but neither $\mathsf{P} \subseteq \mathsf{SKSL}$ nor $\mathsf{P} \subseteq \mathsf{NISMSL}$,
in which case $(\mathsf{SKSL}|\mathsf{NISMSL}) \nsupseteq (\mathsf{P} \cap (\mathsf{NISMSL}|\mathsf{SKSL}))$, and so, by Lemma|Corollary 5.11|5.3 $((\mathfrak{DM}_4 \times \mathfrak{B}_2)|\mathfrak{K}_3) \in \mathsf{P} \ni \mathfrak{S}_3$. Then, by Corollary 5.5, $\mathsf{P} = (\mathsf{SKSL} \cup \mathsf{NISMSL})$.
(3) $\mathsf{P} \subseteq \mathsf{NISMSL}$ but $\mathsf{P} \nsubseteq \mathsf{SKSL}$,
in which case, by Lemma 5.11, $(\mathfrak{DM}_4 \times \mathfrak{B}_2) \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Corollary 5.2, $\mathsf{P} = \mathsf{NISMSL}$.
(4) $\mathsf{P} \subseteq \mathsf{SKSL}$ but $\mathsf{P} \nsubseteq \mathsf{NISMSL}$,
in which case, by Corollary 5.3, $\mathfrak{K}_3 \in \mathsf{P} \ni \mathfrak{S}_3$, and so $\mathsf{P} = \mathsf{SKSL}$.
(5) $\mathsf{P} \subseteq \mathsf{NISKSL}$ but $\mathsf{P} \nsubseteq \mathsf{RSKSL}$,
in which case, by Lemma 5.10, $(\mathfrak{K}_3 \times \mathfrak{B}_2) \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Corollary 5.2, $\mathsf{P} = \mathsf{NISKSL}$.

(6) $\mathsf{P} \subseteq \mathsf{RSKSL}$ but $\mathsf{P} \not\subseteq \mathsf{SL}$,
in which case, by Corollary 5.8, $\mathfrak{K}_4 \in \mathsf{P} \ni \mathfrak{S}_3$, and so, by Theorem 5.9, $\mathsf{P} = \mathsf{RSKSL}$.

(7) $\mathsf{P} \subseteq \mathsf{SL}$,
in which case $\mathsf{P} = \mathsf{SL}$.

In this way, [12, Theorem 4.8] completes the argument. □

This, by Corollaries 4.3, 5.2, 5.5 and Theorem 5.9, immediately yields:

Corollary 5.14. *Any [pre-/-quasi-]variety* $\mathsf{P} \subseteq \mathsf{SMSL}$ *such that* $\mathsf{P} \not\subseteq \mathsf{DML}$ *is generated by* $(\mathsf{P} \cap \mathsf{DML}) \cup \mathsf{SL}$.

6. Conclusions

Perhaps, the most acute problem remained open concerns the lattice of quasivarieties of *all* MS lattices. In this connection, an interesting (though purely methodological) point remained open is whether the *optional* version of Corollary 5.14 can be proved directly *prior* proving Corollaries 4.3, 5.2, 5.5 as well as Theorems 5.9 and 5.13, in which case these would immediately ensue from the main results of [12].

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