On Robin's Criterion for the Riemann Hypothesis

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April 2, 2022

Noname manuscript No.
(will be inserted by the editor)

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the date of receipt and acceptance should be inserted later

Abstract Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 2022, Vega stated that the possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$. Based on that result, we obtain a contradiction just assuming the existence of such possible smallest counterexample $n > 5040$ for the Robin inequality. By contraposition, we show that the Riemann hypothesis should be true.

Keywords Riemann hypothesis · Robin inequality · Sum-of-divisors function · Prime numbers · Counterexample

Mathematics Subject Classification (2010) 11M26 · 11A41 · 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d|n} d$$

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where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [3].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n. We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Theorem 1.2 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [1].*

Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Now, we are able to use this recently result:

Theorem 1.3 *The possible existence of the smallest counterexample* $n > 5040$ *of the Robin inequality implies that* $q_m > e^{31.018189471}$ *and* $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$*, where* $N_m = \prod_{i=1}^{m} q_i$ *is the primorial number of order m,* q_m *is the largest prime divisor of n and* $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ *when n must be an Hardy-Ramanujan integer of the form* $\prod_{i=1}^{m} q_i^{a_i}$ *[4].*

Putting all together yields a proof for the Riemann hypothesis using the Theorem 1.3 as the principal argument.

2 A Central Lemma

These are known results:

Lemma 2.1 *For every* $x > -1$ *[2]:*

$$\log(1+x) \geq \frac{x}{x+1}.$$

Lemma 2.2 *For every real number x [2]:*

$$e^x \geq 1+x.$$

The following is a key Lemma.

Lemma 2.3 *If the natural number* $n > 5040$ *is an Hardy-Ramanujan integer of the form* $\prod_{i=1}^{m} q_i^{a_i}$*, then* $\beta \geq 1 + \sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}$ *where* $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.

Proof If we apply the logarithm to the value of

$$\prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$$

then we obtain that

$$\sum_{i=1}^{m} \log(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}).$$

For some $1 \leq j \leq m$, we know that

$$\frac{q_j^{a_j+1}}{q_j^{a_j+1}-1} = 1 + \frac{1}{q_j^{a_j+1}-1}.$$

We use the Lemma 2.1 to show that

$$\begin{aligned}
\log(1+\frac{1}{q_j^{a_j+1}-1}) &\geq \frac{\frac{1}{q_j^{a_j+1}-1}}{\frac{1}{q_j^{a_j+1}-1}+1} \\
&= \frac{1}{(q_j^{a_j+1}-1)\times(\frac{1}{q_j^{a_j+1}-1}+1)} \\
&= \frac{1}{1+(q_j^{a_j+1}-1)} \\
&= \frac{1}{q_j^{a_j+1}}.
\end{aligned}$$

So,

$$\sum_{i=1}^{m} \log(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}) \geq \sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}$$

and thus,

$$\prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \geq e^{\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}}.$$

Using the Lemma 2.2, we have that

$$e^{\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}} \geq 1 + \sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}$$

and therefore,

$$\beta \geq 1 + \sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}.$$

3 Main Insight

This is the main insight.

Lemma 3.1 *Suppose that* $n > 5040$ *is an Hardy-Ramanujan integer of the form* $\prod_{i=1}^{m} q_i^{a_i}$ *and* $q_m > e^{31.018189471}$. *Then* $(\log n)^{\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}} \geq 1.03352795481$.

Proof If we apply the logarithm to the both sides of the inequality, then

$$\left(\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}\right) \times \log\log n \geq \log(1.03352795481).$$

Let's multiply the both sides of the inequality by e^{γ},

$$\left(\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}\right) \times e^{\gamma} \times \log\log n \geq e^{\gamma} \times \log(1.03352795481).$$

From the Theorem 1.2, we know that

$$\begin{aligned} e^{\gamma} \times \log\log n &\geq e^{\gamma} \times \log\log N_m \\ &> f(N_m) \\ &= \prod_{i=1}^{m}(1+\frac{1}{q_i}) \end{aligned}$$

since $n > 5040$ is an Hardy-Ramanujan integer, $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and thus, $n \geq N_m$ and N_m is square free. Hence, we would have that

$$\left(\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}\right) \times \prod_{i=1}^{m}(1+\frac{1}{q_i}) \geq e^{\gamma} \times \log(1.03352795481).$$

If we apply the logarithm to the both sides again, then

$$\log\left(\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}\right) + \sum_{i=1}^{m} \log(1+\frac{1}{q_i}) \geq \log(e^{\gamma} \times \log(1.03352795481)).$$

We use the Lemma 2.1 to show that

$$\begin{aligned} \log\left(\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}\right) &= \log\left(1+(-1+\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}})\right) \\ &\geq \frac{(-1+\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}})}{(-1+\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}})+1} \\ &= \frac{(-1+\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}})}{\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}} \\ &= 1-\frac{1}{\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}} \end{aligned}$$

since

$$-1+\sum_{i=1}^{m}\frac{1}{q_i^{a_i+1}} > -1.$$

For some $1 \leq j \leq m$, we know that

$$\begin{aligned}\log(1+\frac{1}{q_j}) &\geq \frac{\frac{1}{q_j}}{\frac{1}{q_j}+1}\\ &= \frac{1}{q_j \times (\frac{1}{q_j}+1)}\\ &= \frac{1}{1+q_j}\end{aligned}$$

according to the Lemma 2.1. However, we note that

$$1-\frac{1}{\sum_{i=1}^{m}\frac{1}{q_i^{a_i+1}}}+\sum_{i=1}^{m}\frac{1}{1+q_i} > 0$$

when $q_m > e^{31.018189471}$. In addition, we have that

$$0 > \log(e^{\gamma} \times \log(1.03352795481))$$

and finally, the proof is complete.

4 Main Theorem

We conclude with the following statement:

Theorem 4.1 *The Riemann hypothesis is true.*

Proof Suppose that $n > 5040$ is the possible smallest number such that $\mathsf{Robins}(n)$ does not hold. By the Theorem 1.3, we know that $q_m > e^{31.018189471}$ and $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^{m}\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$. From the Lemma 2.3, we know that

$$(\log n)^{\beta} \geq (\log n)^{\left(1+\sum_{i=1}^{m}\frac{1}{q_i^{a_i+1}}\right)}$$

and therefore, we would have that

$$(\log n)^{\left(1+\sum_{i=1}^{m}\frac{1}{q_i^{a_i+1}}\right)} < 1.03352795481 \times \log(N_m)$$

when $n > 5040$ is the possible smallest number such that $\mathsf{Robins}(n)$ does not hold. Thus, we would obtain that

$$(\log n)^{\sum_{i=1}^{m} \frac{1}{q_i^{a_i+1}}} < 1.03352795481$$

since n must be an Hardy-Ramanujan integer and so, $\log n \geq \log N_m$. However, we know the previous inequality cannot be satisfied because of the Lemma 3.1. By contraposition, we show that the Riemann hypothesis is true, since we obtain a contradiction just assuming the possible smallest counterexample for the Robin inequality greater than 5040. Certainly, this is a direct consequence of the Theorem 1.1.

Acknowledgments

The author would like to thank his mother, maternal brother and his friend Sonia for their support.

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